Pre-Calculus 12 – Comprehensive Review Worksheet

Detailed Answer Key Included

Worksheet

Part A – Functions, Relations & Conics

1. Determine the domain and range of the function

$$f(x) = \frac{\sqrt{8 - 3x}}{x - 5}.$$

2. Given the parabola $g(x) = -4(x + 3)^2 + 7$, state its vertex, axis of symmetry, direction of opening, and range.

3. If $f^{-1}(x) = \frac{5x-4}{3}$, find $f(x)$.

4. Convert the general form conic

$$4x^2 - 9y^2 + 24x - 36y - 36 = 0$$

into standard form, identify the conic, and list its key properties (center, vertices, foci, asymptotes if applicable).

Part B – Exponents & Logarithms

5. Solve for x:

$$3^{2x-3} = 81.$$

6. Simplify as a single logarithm:

$$\log_7 49 + \frac{1}{2} \log_7 7.$$

7. Solve the equation:

$$\log_4(x + 1) - \log_4(x - 3) = 2.$$

Part C – Polynomial & Rational Functions

8. Use synthetic division to test whether $x-2$ is a factor of

$$p(x) = x^5 - 4x^4 + x^3 + 6x^2 - 8x + 4.$$

Then find all real zeros of $p(x)$ (approximations are acceptable).

9. Sketch and analyze

$$y = \frac{(x-2)^2(x+1)}{(x-3)(x+4)^2}.$$

Include vertical asymptotes, horizontal asymptotes, zeros, y-intercept, and sign behavior.

Part D – Sequences & Series

10. A ball is dropped from a height of 10 meters and bounces to 60% of its previous height each time. Calculate the total vertical distance traveled by the ball after it hits the ground for the 5th time.

11. Given the sum of the first n terms of an arithmetic sequence is

$$S_n = 7n^2 + 2n,$$

find the first three terms a_1, a_2, a_3 and give a formula for the general term a_n.

Part E – Trigonometry & Transformations

12. Solve for θ in degrees, where $0^\circ \le \theta < 360^\circ$:

$$4\sin\theta - 2 = 0.$$

13. Verify the identity:

$$\frac{1+\cos 2x}{\sin 2x} = \cot x,$$

and state the domain restrictions.

14. A Ferris wheel has radius 20 m and takes 50 seconds to complete one full rotation. The bottom of the wheel is 3 meters above the ground. A rider boards at the bottom at time $t = 0$.

(a) Write a sinusoidal model for the rider's height $h(t)$ above the ground in meters.

(b) Calculate the rider's height after 20 seconds.

15. Describe the amplitude, period, phase shift, and vertical shift of the function

$$y = 3\cos\left(2x - \frac{\pi}{4}\right) + 5.$$

Detailed Answer Key (Step-by-step)

Part A – Functions, Relations & Conics

1. $f(x) = \frac{\sqrt{8-3x}}{x-5}$

- Radicand: $8 - 3x \geq 0 \Rightarrow x \leq \frac{8}{3} \approx 2.6667$.
- Denominator: $x \neq 5$.
- Since $5 > 2.6667$, denominator zero is outside domain.
- Domain: $\boxed{(-\infty, \frac{8}{3}]}$.
- Range: numerator ≥ 0, denominator < 0 for $x \leq 2.6667 < 5$, so function is negative or zero. Function tends to 0 from below as $x \to -\infty$ and becomes large negative near $x = 5$, but 5 is outside domain, so no vertical asymptote.
- Range is $\boxed{(-\infty, 0]}$.

2. $g(x) = -4(x+3)^2 + 7$

- Vertex: $(-3, 7)$.
- Axis of symmetry: $x = -3$.
- Opens downward (coefficient $-4 < 0$).
- Range: $y \leq 7$.

3. Given $f^{-1}(x) = \frac{5x-4}{3}$,

$$y = \frac{5x-4}{3} \implies x = \frac{5y-4}{3} \implies 3x = 5y - 4 \implies 5y = 3x + 4 \implies y = \frac{3x+4}{5}.$$

$$\boxed{f(x) = \frac{3x+4}{5}.}$$

4. Convert

$$4x^2 - 9y^2 + 24x - 36y - 36 = 0$$

to standard form:

Group x and y terms:

$$4(x^2 + 6x) - 9(y^2 + 4y) = 36.$$

Complete the square:

$$x^2 + 6x + 9 - 9 = (x+3)^2 - 9,$$
$$y^2 + 4y + 4 - 4 = (y+2)^2 - 4.$$

Rewrite:

$$4[(x+3)^2 - 9] - 9[(y+2)^2 - 4] = 36,$$
$$4(x+3)^2 - 36 - 9(y+2)^2 + 36 = 36,$$
$$4(x+3)^2 - 9(y+2)^2 = 36.$$

Divide both sides by 36:

$$\frac{(x+3)^2}{9} - \frac{(y+2)^2}{4} = 1.$$

This is a hyperbola centered at $(-3, -2)$ with:

$$a^2 = 9, \quad b^2 = 4.$$

Vertices:

$$(-3 \pm 3, -2) \Rightarrow (0, -2) \text{ and } (-6, -2).$$

Foci distance:

$$c = \sqrt{a^2 + b^2} = \sqrt{9+4} = \sqrt{13} \approx 3.606.$$

Foci:

$$(-3 \pm 3.606, -2).$$

Asymptotes:

$$y + 2 = \pm\frac{b}{a}(x+3) = \pm\frac{2}{3}(x+3).$$

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Part B – Exponents & Logarithms

5. Solve $3^{2x-3} = 81$.
Rewrite $81 = 3^4$.

$$2x - 3 = 4 \implies 2x = 7 \implies x = \frac{7}{2} = 3.5.$$

$$\boxed{x = 3.5}.$$

6. Simplify $\log_7 49 + \frac{1}{2}\log_7 7$.

$$\log_7 49 = 2, \quad \log_7 7 = 1.$$

$$2 + \frac{1}{2} \times 1 = 2 + \frac{1}{2} = \frac{5}{2}.$$

As a single logarithm:

$$\boxed{\log_7 7^{5/2} = \log_7 \sqrt{7^5} = \log_7(7^{2.5})}.$$

7. Solve

$$\log_4(x+1) - \log_4(x-3) = 2.$$

Use log subtraction:

$$\log_4 \frac{x+1}{x-3} = 2 \implies \frac{x+1}{x-3} = 4^2 = 16.$$

Solve:

$$x + 1 = 16(x-3) \implies x + 1 = 16x - 48 \implies -15x = -49 \implies x = \frac{49}{15} \approx 3.2667.$$

Domain check:

$$x + 1 > 0 \implies x > -1, \quad x - 3 > 0 \implies x > 3.$$

$x = \frac{49}{15} \approx 3.2667 > 3$, valid.

$$\boxed{x = \frac{49}{15}}.$$

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Part C – Polynomial & Rational Functions

8. Synthetic division of $p(x) = x^5 - 4x^4 + x^3 + 6x^2 - 8x + 4$ by $x - 2$:
Coefficients: $1, -4, 1, 6, -8, 4$

$$\begin{array}{c|cccccc} 2 & 1 & -4 & 1 & 6 & -8 & 4 \\ & & 2 & -4 & -6 & 0 & -16 \\ \hline & 1 & -2 & -3 & 0 & -8 & -12 \end{array}$$

Remainder: $-12 \neq 0$, so $x - 2$ is not a factor.
Using numerical methods (calculator or software) approximate zeros:
- Real zeros approximately at: $3.69, 1.12, -1.87$. - The other two zeros are complex.
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9. For

$$y = \frac{(x-2)^2(x+1)}{(x-3)(x+4)^2},$$

- Vertical asymptotes: $x = 3$ (simple), $x = -4$ (multiplicity 2).
- Horizontal asymptote: degrees numerator and denominator both 3, leading coefficients ratio is $1/1 = 1$, so $y = 1$.
- Zeros: $x = 2$ (multiplicity 2, touches and bounces), $x = -1$ (simple zero).
- y-intercept: at $x = 0$,

$$y(0) = \frac{(0-2)^2(0+1)}{(0-3)(0+4)^2} = \frac{4 \times 1}{-3 \times 16} = -\frac{4}{48} = -\frac{1}{12}.$$

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Part D – Sequences & Series

10. Bouncing ball:

- Initial drop: 10 m. - Each bounce height: 60% of previous. - Distance after 5th ground hit = initial drop + 2× sum of first 4 bounce heights.

Bounce heights form geometric sequence:

$$a_1 = 10 \times 0.6 = 6, \quad r = 0.6.$$

Sum of first 4 bounces:

$$S_4 = 6\frac{1-0.6^4}{1-0.6} = 6\frac{1-0.1296}{0.4} = 6 \times \frac{0.8704}{0.4} = 6 \times 2.176 = 13.056.$$

Total distance:

$$D = 10 + 2 \times 13.056 = 10 + 26.112 = 36.112 \text{ meters}.$$

$$\boxed{36.11 \text{ meters (approx.)}}.$$

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11. Given

$$S_n = 7n^2 + 2n,$$

find a_1, a_2, a_3 and formula for a_n.

$$a_1 = S_1 = 7 + 2 = 9.$$

$$a_n = S_n - S_{n-1} = [7n^2 + 2n] - [7(n-1)^2 + 2(n-1)].$$

Expand:

$$= 7n^2 + 2n - [7(n^2 - 2n + 1) + 2n - 2] = 7n^2 + 2n - [7n^2 - 14n + 7 + 2n - 2].$$

Simplify bracket:

$$7n^2 + 2n - 7n^2 + 14n - 7 - 2n + 2 = (2n + 14n - 2n) + (-7 + 2) = 14n - 5.$$

$$a_n = 14n - 5.$$

Check first three terms:

$$a_1 = 14(1) - 5 = 9, \quad a_2 = 14(2) - 5 = 23, \quad a_3 = 14(3) - 5 = 37.$$

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Part E – Trigonometry & Transformations

12. Solve

$$4\sin\theta - 2 = 0 \implies \sin\theta = \frac{1}{2}.$$

Solutions for θ in $[0^\circ, 360^\circ)$:

$$\theta = 30^\circ, 150^\circ.$$

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13. Verify

$$\frac{1 + \cos 2x}{\sin 2x} = \cot x.$$

Use identities:

$$1 + \cos 2x = 2\cos^2 x, \quad \sin 2x = 2\sin x \cos x,$$

so

$$\frac{2\cos^2 x}{2\sin x \cos x} = \frac{\cos x}{\sin x} = \cot x.$$

Domain restriction: $\sin 2x \neq 0 \implies 2x \neq k\pi \implies x \neq \frac{k\pi}{2}$ for integer k.

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14. Ferris wheel: radius 20 m, period 50 s, bottom 3 m above ground, rider boards at bottom at $t = 0$.

- Center height: $3 + 20 = 23$ m.
- Amplitude: 20 m.
- Angular velocity:

$$\omega = \frac{2\pi}{50} = \frac{\pi}{25} \text{ rad/s}.$$

- Rider starts at lowest point, so use a cosine function shifted:

$$h(t) = 23 - 20\cos\left(\frac{\pi}{25}t\right).$$

(a) Model:

$$\boxed{h(t) = 23 - 20\cos\left(\frac{\pi}{25}t\right)}.$$

(b) Height at $t = 20$ s:

$$h(20) = 23 - 20\cos\left(\frac{\pi}{25} \times 20\right) = 23 - 20\cos\left(\frac{20\pi}{25}\right) = 23 - 20\cos\left(\frac{4\pi}{5}\right).$$

$$\cos\left(\frac{4\pi}{5}\right) = \cos 144^\circ = -0.8090,$$

so

$$h(20) = 23 - 20 \times (-0.8090) = 23 + 16.18 = 39.18 \text{ m}.$$

$$\boxed{39.18 \text{ meters (approx.)}}.$$

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15. For

$$y = 3\cos\left(2x - \frac{\pi}{4}\right) + 5,$$

- Amplitude: 3.
- Period: $\frac{2\pi}{2} = \pi$.
- Phase shift: Solve $2x - \frac{\pi}{4} = 0 \implies x = \frac{\pi}{8}$ (right by $\frac{\pi}{8}$).
- Vertical shift: 5.

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