Pre-Calculus 12 – Comprehensive Review Worksheet (Corrected)

Detailed Answer Key Included

Worksheet

Part A – Functions & Relations

1. For the function $f(x) = \dfrac{\sqrt{5-x}}{x-3}$, find the **domain** and **range**.

2. For the function $g(x) = -2(x+1)^2 + 8$, find the **vertex**, **axis of symmetry**, and **range**. Also state whether the parabola opens upward or downward.

3. Given the inverse function $f^{-1}(x) = \dfrac{x-5}{2}$, find the original function $f(x)$.

Part B – Exponents & Logarithms

4. Solve for x: $5^{2x-1} = 7$.

5. Simplify the expression $\log_3 81 - \frac{1}{2}\log_3 9$ as a single logarithm.

6. Solve for x: $\log_2(x+3) + \log_2(x-1) = 4$. State the domain restrictions and check which solutions are valid.

Part C – Polynomial & Rational Functions

7. For the polynomial $p(x) = 2x^4 - 3x^3 - 11x^2 + 6x + 18$:

(a) Test whether $x - 2$ is a factor by evaluating $p(2)$.

(b) Find all zeros of the polynomial, approximating decimal values if necessary.

8. Sketch or describe the rational function:

$$y = \frac{(x-2)(x+1)^2}{(x-3)^2(x+4)}.$$

Identify and state the **vertical asymptotes**, **horizontal asymptote**, **zeros (x-intercepts)**, **y-intercept**, and describe the sign (positive/negative) of the function in each interval divided by these critical points.

Part D – Sequences & Series

9. A geometric sequence has first term $a_1 = 8$ and fourth term $a_4 = 1$. Find:

(a) The common ratio r.

(b) The sum of the first 10 terms, S_{10}.

10. Given the sum of the first n terms of an arithmetic sequence is $S_n = 4n^2 - n$:

(a) Find the first three terms a_1, a_2, a_3.

(b) Derive a formula for the n^{th} term a_n.

Part E – Trigonometry

11. Solve the equation $2\sin\theta - 1 = 0$ for $0^\circ \le \theta < 360^\circ$.

12. Verify the trigonometric identity:

$$\frac{1 - \cos 2x}{\sin 2x} = \tan x.$$

State any domain restrictions necessary.

13. A Ferris wheel has a radius of $R = 15$ m and completes one revolution every $T = 40$ seconds. The bottom of the wheel is 2 m above the ground. A rider boards at the bottom at time $t = 0$.

(a) Write a sinusoidal function $h(t)$ that models the rider's height above the ground at time t.

(b) Calculate the rider's height at $t = 25$ seconds.

Part F – Mixed Challenge (Projectile)

14. A ball is thrown from the top of a building that is 45 m high, with an initial speed of 20 m/s at an angle of 30° above the horizontal. Use $g = 9.8$ m/s^2.

(a) Calculate the time it takes for the ball to hit the ground.

(b) Calculate the horizontal distance the ball travels before hitting the ground.

Detailed Answer Key (Step-by-step)

Part A – Functions & Relations

1. $f(x) = \dfrac{\sqrt{5-x}}{x-3}$

- **Domain:** The radicand requires $5 - x \geq 0 \Rightarrow x \leq 5$. The denominator requires $x \neq 3$. Combine:

$$\boxed{\text{Domain: } \{x \mid x \leq 5,\ x \neq 3\}.}$$

- **Range:** Behavior near the vertical asymptote $x = 3$:
  - For $x \to 3^-$, numerator $\sqrt{5-x} \to \sqrt{2} > 0$ and denominator $\to 0^-$, so $f(x) \to -\infty$.
  - For $x \to 3^+$, denominator $\to 0^+$, so $f(x) \to +\infty$.

  The function covers all real numbers:

$$\boxed{\text{Range: } \mathbb{R}.}$$

2. $g(x) = -2(x+1)^2 + 8$

- Vertex is at $(-1, 8)$.
- Axis of symmetry: $x = -1$.
- Parabola opens downward (coefficient $-2 < 0$).
- Range: $y \leq 8$.

3. Given $f^{-1}(x) = \dfrac{x-5}{2}$, find $f(x)$.

$$x = \frac{y-5}{2} \implies y = 2x + 5,$$

so

$$\boxed{f(x) = 2x + 5.}$$

Part B – Exponents & Logarithms

4. Solve $5^{2x-1} = 7$:

$$2x - 1 = \log_5 7 \implies x = \frac{1 + \log_5 7}{2} \approx 1.10453.$$

5. Simplify $\log_3 81 - \frac{1}{2}\log_3 9$:

$$\log_3 81 = 4, \quad \tfrac{1}{2}\log_3 9 = 1,$$

so the expression equals

$$4 - 1 = 3 = \log_3 27.$$

6. Solve $\log_2(x+3) + \log_2(x-1) = 4$:

$$\log_2[(x+3)(x-1)] = 4 \implies (x+3)(x-1) = 16.$$

Expanding:

$$x^2 + 2x - 3 = 16 \implies x^2 + 2x - 19 = 0.$$

Quadratic formula:

$$x = -1 \pm 2\sqrt{5}.$$

Domain restrictions: $x > 1$ (from $x - 1 > 0$), so

$$\boxed{x = -1 + 2\sqrt{5} \approx 3.4721.}$$

Part C – Polynomial & Rational Functions

7. Polynomial $p(x) = 2x^4 - 3x^3 - 11x^2 + 6x + 18$

(a) Test $x = 2$:

$$p(2) = 2(16) - 3(8) - 11(4) + 6(2) + 18 = -6 \neq 0,$$

so $x - 2$ is not a factor.

(b) Approximate zeros (using numeric methods):

$$x \approx 2.666, \quad 1.613, \quad -1.390 \pm 0.402i.$$

8. Rational function:

$$y = \frac{(x-2)(x+1)^2}{(x-3)^2(x+4)}.$$

- Vertical asymptotes: $x = 3$ (double), $x = -4$ (simple).
- Horizontal asymptote: degree numerator = degree denominator = 3, leading coefficient ratio = 1, so $y = 1$.
- Zeros at $x = 2$ (simple), $x = -1$ (double).
- y-intercept at $x = 0$:

$$y(0) = \frac{(-2)(1)^2}{(-3)^2(4)} = -\frac{1}{18}.$$

- Sign chart: analyze intervals divided by $-4, -1, 2, 3$ to determine where y is positive or negative.

Part D – Sequences & Series

9. Geometric sequence: $a_1 = 8$, $a_4 = 1$.

$$1 = 8r^3 \implies r = \frac{1}{2}.$$

Sum of first 10 terms:

$$S_{10} = 8 \cdot \frac{1 - (1/2)^{10}}{1 - 1/2} = \frac{1023}{64} \approx 15.984.$$

10. Sum of arithmetic sequence: $S_n = 4n^2 - n$.

$$\begin{aligned} a_1 &= S_1 = 3, \\ a_n &= S_n - S_{n-1} = (4n^2 - n) - [4(n-1)^2 - (n-1)] = 8n - 5. \end{aligned}$$

So

$$a_1 = 3, \quad a_2 = 11, \quad a_3 = 19,$$

and

$$a_n = 8n - 5.$$

Part E – Trigonometry

11. Solve $2\sin\theta - 1 = 0$:

$$\sin\theta = \frac{1}{2} \implies \theta = 30^\circ, 150^\circ.$$

12. Verify identity:

$$\frac{1 - \cos 2x}{\sin 2x} = \tan x.$$

Using double-angle formulas:

$$1 - \cos 2x = 2\sin^2 x, \quad \sin 2x = 2\sin x \cos x,$$

so

$$\frac{1 - \cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin x \cos x} = \frac{\sin x}{\cos x} = \tan x.$$

Domain restriction: $\sin 2x \neq 0 \implies x \neq k\frac{\pi}{2}$ for integer k.

13. Ferris wheel problem:

- Center height $= 2 + 15 = 17$ m, amplitude $= 15$.

- Angular speed $\omega = \frac{2\pi}{40} = \frac{\pi}{20}$ rad/s.
- Rider starts at bottom at $t = 0$, so model:

$$h(t) = 17 - 15\cos\left(\frac{\pi}{20}t\right).$$

- Height at $t = 25$ s:

$$h(25) = 17 - 15\cos\left(\frac{5\pi}{4}\right) = 17 + \frac{15\sqrt{2}}{2} \approx 27.61 \text{ m}.$$

Part F – Mixed Challenge (Projectile)

Initial speed $v = 20$ m/s at $30°$, initial height $y_0 = 45$ m, $g = 9.8$ m/s^2.

- Horizontal velocity:

$$v_x = 20\cos 30° = 10\sqrt{3} \approx 17.32 \text{ m/s}.$$

- Vertical velocity:

$$v_{y0} = 20\sin 30° = 10 \text{ m/s}.$$

- Vertical position function:

$$y(t) = 45 + 10t - 4.9t^2.$$

- Solve $y(t) = 0$:

$$-4.9t^2 + 10t + 45 = 0.$$

  Positive root:

$$t = \frac{-10 + \sqrt{100 + 882}}{-9.8} \approx 4.22 \text{ s}.$$

- Horizontal distance:

$$d = v_x \cdot t \approx 17.32 \times 4.22 = 73.06 \text{ m}.$$

Sources & Notes

- Stewart, Redlin, Watson, *Precalculus: Mathematics for Calculus*, 7th Edition.
- OpenStax, *Precalculus*, 2016. https://openstax.org/details/books/precalculus
- Serway, *Physics for Scientists and Engineers*, for projectile motion basics.