Angular Measure
ApexMath

Notes

Two Ways to Measure Angles

So far, we have used **degrees** to measure angles. In Pre-Calculus we introduce a second system called **radians**. Both systems describe the exact same rotations — they are simply different units, like centimetres and inches measuring the same length.

What Is a Radian?

A radian is defined using the circle itself. If you take the radius of a circle and wrap that length along the circumference, the arc it covers subtends an angle of exactly **1 radian** at the centre.

Because the full circumference of a circle with radius r is $2\pi r$, one complete rotation uses up 2π radius-lengths. This gives the key conversion fact:

$$360^\circ = 2\pi \text{ rad} \qquad \Longrightarrow \qquad 180^\circ = \pi \text{ rad}$$

Converting Between Degrees and Radians

Starting from $180^\circ = \pi$, divide both sides to produce a conversion factor.

$$\text{Degrees} \to \text{Radians:} \qquad \theta_{\text{rad}} = \theta_{\text{deg}} \times \frac{\pi}{180^\circ}$$

$$\text{Radians} \to \text{Degrees:} \qquad \theta_{\text{deg}} = \theta_{\text{rad}} \times \frac{180^\circ}{\pi}$$

Common Angle Conversions to Memorise

Degrees	**Radians**
30°	$\frac{\pi}{6}$
45°	$\frac{\pi}{4}$
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$
180°	π
270°	$\frac{3\pi}{2}$
360°	2π

Arc Length

When a central angle θ (in **radians**) is drawn inside a circle of radius r, the arc it cuts off has length:

$$s = r\theta$$

Both r and s use the same unit of length, and θ **must be in radians**.

Area of a Sector

A sector is the pie-slice region of a circle bounded by two radii and an arc. Its area is:

$$A = \frac{1}{2}r^2\theta$$

where again θ must be in radians.

Pro Tip

- Think of $\frac{\pi}{180}$ as a multiplier. Multiply by it to go to radians; multiply by $\frac{180}{\pi}$ to go back to degrees.
- Answers that contain π are **exact**. Only switch to a decimal approximation when the problem specifically asks for one.
- The arc length and sector area formulas **only work with radians**. If the angle is given in degrees, convert it first — every single time, without exception.
- A quick sanity check: $\pi \approx 3.14$, so $\frac{\pi}{2} \approx 1.57$ and $\frac{\pi}{4} \approx 0.79$. Use these rough values to catch conversion errors.

Examples

Example 1: Convert $120°$ to radians.

Step 1: Multiply by the conversion factor $\frac{\pi}{180°}$.

$$120° \times \frac{\pi}{180°} = \frac{120\pi}{180}$$

Step 2: Simplify the fraction. Both 120 and 180 are divisible by 60.

$$\frac{120}{180} = \frac{2}{3} \qquad \Longrightarrow \qquad 120° = \frac{2\pi}{3} \text{ rad}$$

Example 2: Convert $\frac{5\pi}{4}$ radians to degrees.

Step 1: Multiply by the conversion factor $\frac{180°}{\pi}$.

$$\frac{5\pi}{4} \times \frac{180°}{\pi}$$

Step 2: Cancel π and simplify.

$$= \frac{5 \times 180^\circ}{4} = \frac{900^\circ}{4} = 225^\circ$$

Example 3: Find the arc length for a circle with radius $r = 5$ cm and central angle $\theta = \frac{\pi}{3}$.

Step 1: Confirm that θ is already in radians. It is.

Step 2: Apply the arc length formula.

$$s = r\theta = 5 \cdot \frac{\pi}{3} = \frac{5\pi}{3} \text{ cm}$$

Example 4: Find the area of a sector with radius $r = 4$ cm and central angle $\theta = \frac{\pi}{4}$.

Step 1: Apply the sector area formula.

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}(4)^2 \cdot \frac{\pi}{4}$$

Step 2: Evaluate step by step. Square the radius first, then multiply.

$$= \frac{1}{2} \cdot 16 \cdot \frac{\pi}{4} = \frac{16\pi}{8} = 2\pi \text{ cm}^2$$

Common Mistakes

- **Forgetting to convert to radians before using the formulas.** Arc length and sector area both require θ in radians. If the angle is given in degrees, convert it first — then substitute into the formula.
- **Cancelling π incorrectly.** When you multiply a radian measure by $\frac{180^\circ}{\pi}$, only the π in the numerator of the radian expression cancels with the π in the denominator of the conversion factor. Nothing else cancels.
- **Leaving answers as decimals when exact form is expected.** Keep π in your answer unless the problem explicitly asks you to evaluate numerically.
- **Mixing up arc length and sector area.** Arc length $s = r\theta$ is a length (one dimension). Sector area $A = \frac{1}{2}r^2\theta$ is an area (two dimensions). Check the units in your answer — they reveal which formula was used.

Worksheet

1. Convert each angle from degrees to radians. Leave your answer in terms of π.

   (a) 30°

(b) $150°$

(c) $270°$

(d) $315°$

2. Convert each angle from radians to degrees.

   (a) $\dfrac{\pi}{6}$

   (b) $\dfrac{3\pi}{4}$

   (c) $\dfrac{7\pi}{6}$

   (d) 2π

3. Find the arc length for each circle. Leave answers in terms of π where possible.

   (a) $r = 3$ cm, $\quad \theta = \dfrac{\pi}{2}$

   (b) $r = 10$ m, $\quad \theta = \dfrac{2\pi}{5}$

   (c) $r = 6$ cm, $\quad \theta = 60°$ *(Convert the angle first.)*

4. Find the area of each sector.

   (a) $r = 5$ cm, $\quad \theta = \dfrac{\pi}{3}$

   (b) $r = 8$ m, $\quad \theta = \dfrac{3\pi}{4}$

5. A wheel of radius 12 cm rotates through an angle of $\frac{5\pi}{6}$ radians. How long is the arc traced by a point on the rim?

6. A pizza slice has a crust length (arc) of 6π cm and a radius of 9 cm. Find the central angle of the slice in radians, then convert it to degrees.

Answer Key

1. Degrees to Radians.

(a) $30°$

Step 1: Multiply by $\frac{\pi}{180°}$.

$$30° \times \frac{\pi}{180°} = \frac{30\pi}{180} = \frac{\pi}{6}$$

$$\boxed{\frac{\pi}{6}}$$

(b) $150°$

Step 1: Multiply by $\frac{\pi}{180°}$.

$$150° \times \frac{\pi}{180°} = \frac{150\pi}{180} = \frac{5\pi}{6}$$

$$\boxed{\frac{5\pi}{6}}$$

(c) $270°$

Step 1: Multiply by $\frac{\pi}{180°}$.

$$270° \times \frac{\pi}{180°} = \frac{270\pi}{180} = \frac{3\pi}{2}$$

$$\boxed{\frac{3\pi}{2}}$$

(d) $315°$

Step 1: Multiply by $\frac{\pi}{180°}$.

$$315° \times \frac{\pi}{180°} = \frac{315\pi}{180} = \frac{7\pi}{4}$$

$$\boxed{\frac{7\pi}{4}}$$

2. Radians to Degrees.

(a) $\frac{\pi}{6}$

Step 1: Multiply by $\frac{180°}{\pi}$. The π terms cancel.

$$\frac{\pi}{6} \times \frac{180°}{\pi} = \frac{180°}{6} = 30°$$

$$\boxed{30°}$$

(b) $\dfrac{3\pi}{4}$

Step 1: Multiply by $\dfrac{180^\circ}{\pi}$.

$$\frac{3\pi}{4} \times \frac{180^\circ}{\pi} = \frac{3 \times 180^\circ}{4} = \frac{540^\circ}{4} = 135^\circ$$

$$\boxed{135^\circ}$$

(c) $\dfrac{7\pi}{6}$

Step 1: Multiply by $\dfrac{180^\circ}{\pi}$.

$$\frac{7\pi}{6} \times \frac{180^\circ}{\pi} = \frac{7 \times 180^\circ}{6} = \frac{1260^\circ}{6} = 210^\circ$$

$$\boxed{210^\circ}$$

(d) 2π

Step 1: Multiply by $\dfrac{180^\circ}{\pi}$.

$$2\pi \times \frac{180^\circ}{\pi} = 2 \times 180^\circ = 360^\circ$$

$$\boxed{360^\circ}$$

3. Arc Length.

(a) $r = 3$ cm, $\theta = \dfrac{\pi}{2}$

Step 1: Apply $s = r\theta$.

$$s = 3 \cdot \frac{\pi}{2} = \frac{3\pi}{2} \text{ cm}$$

$$\boxed{s = \frac{3\pi}{2} \text{ cm}}$$

(b) $r = 10$ m, $\theta = \dfrac{2\pi}{5}$

Step 1: Apply $s = r\theta$.

$$s = 10 \cdot \frac{2\pi}{5} = \frac{20\pi}{5} = 4\pi \text{ m}$$

$$\boxed{s = 4\pi \text{ m}}$$

(c) $r = 6$ cm, $\theta = 60^\circ$

Step 1: Convert 60° to radians.

$$60^\circ \times \frac{\pi}{180^\circ} = \frac{60\pi}{180} = \frac{\pi}{3}$$

Step 2: Apply $s = r\theta$.

$$s = 6 \cdot \frac{\pi}{3} = \frac{6\pi}{3} = 2\pi \text{ cm}$$

$$\boxed{s = 2\pi \text{ cm}}$$

4. Sector Area.

(a) $r = 5$ cm, $\theta = \dfrac{\pi}{3}$

Step 1: Apply $A = \dfrac{1}{2}r^2\theta$.

$$A = \frac{1}{2}(5)^2 \cdot \frac{\pi}{3} = \frac{1}{2} \cdot 25 \cdot \frac{\pi}{3} = \frac{25\pi}{6} \text{ cm}^2$$

$$\boxed{A = \frac{25\pi}{6} \text{ cm}^2}$$

(b) $r = 8$ m, $\theta = \dfrac{3\pi}{4}$

Step 1: Apply $A = \dfrac{1}{2}r^2\theta$.

$$A = \frac{1}{2}(8)^2 \cdot \frac{3\pi}{4} = \frac{1}{2} \cdot 64 \cdot \frac{3\pi}{4} = 32 \cdot \frac{3\pi}{4} = \frac{96\pi}{4} = 24\pi \text{ m}^2$$

$$\boxed{A = 24\pi \text{ m}^2}$$

5. $r = 12$ cm, $\theta = \dfrac{5\pi}{6}$.

Step 1: Apply $s = r\theta$.

$$s = 12 \cdot \frac{5\pi}{6} = \frac{60\pi}{6} = 10\pi \text{ cm}$$

$$\boxed{s = 10\pi \text{ cm}}$$

6. $s = 6\pi$ cm, $r = 9$ cm. Find θ.

Step 1: Start from $s = r\theta$ and isolate θ.

$$\theta = \frac{s}{r} = \frac{6\pi}{9} = \frac{2\pi}{3} \text{ rad}$$

Step 2: Convert to degrees.

$$\frac{2\pi}{3} \times \frac{180^\circ}{\pi} = \frac{2 \times 180^\circ}{3} = \frac{360^\circ}{3} = 120^\circ$$

$$\boxed{\theta = \frac{2\pi}{3} \text{ rad} = 120^\circ}$$

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