Lesson 8: Applications of Trigonometric Functions
ApexMath

Notes

Trigonometric functions are not just abstract math — they show up in the real world whenever something repeats in a regular cycle. Tides rise and fall. Ferris wheels go around. Temperature changes from summer to winter and back. In this lesson, you will learn how to build and use equations that describe these real-world patterns.

Part 1: The Sinusoidal Model

Any situation that repeats in a smooth, regular cycle can be modelled by either a sine or cosine function in the form:

$$y = a\sin\big(b(x-c)\big) + d \qquad \text{or} \qquad y = a\cos\big(b(x-c)\big) + d$$

Each of the four constants a, b, c, and d controls a different feature of the graph. Here is what each one means:

a — **Amplitude**
The amplitude is how far the graph reaches above (or below) the midline.

$$\text{Amplitude} = |a|$$

If a is negative, the graph is flipped vertically (reflected over the midline).
A large amplitude means the wave is tall. A small amplitude means it is flat.
You can also calculate it directly from real data:

$$a = \frac{\text{maximum value} - \text{minimum value}}{2}$$

b — **Controls the Period**
The period is how long it takes to complete one full cycle.

$$\text{Period} = \frac{2\pi}{b} \qquad \Longrightarrow \qquad b = \frac{2\pi}{\text{Period}}$$

A larger value of b makes the wave cycle faster (shorter period). A smaller value of b makes it cycle more slowly (longer period).

c — **Phase Shift (Horizontal Shift)**
The phase shift moves the graph left or right.

- If $c > 0$: the graph shifts **right** by c units.

- If $c < 0$: the graph shifts **left** by $|c|$ units.

In applications, the phase shift tells you *when* the pattern begins. For example, if a tide reaches its maximum at $t = 2$ hours, that is a phase shift of $c = 2$.

d — **Vertical Shift (Midline)**

The vertical shift moves the entire graph up or down. It is the equation of the horizontal line that runs through the middle of the wave — the **midline**.

$$d = \frac{\text{maximum value} + \text{minimum value}}{2}$$

Summary Table

Constant	**What It Controls**	**How to Find It**
a	Amplitude	$\frac{\max - \min}{2}$
b	Period	$b = \frac{2\pi}{\text{Period}}$
c	Horizontal shift	Location of the first max or min
d	Midline / vertical shift	$\frac{\max + \min}{2}$

Choosing Sine vs. Cosine

Both sine and cosine can model the same situation — you just need to handle the phase shift differently. However, there are two shortcuts that simplify things:

- Use **cosine** (with possibly a negative a) when the problem starts at a **maximum or minimum**. Unshifted cosine begins at its maximum, and $-\cos$ begins at its minimum.
- Use **sine** when the problem starts at the **midline**. Unshifted sine begins at zero (the midline).

In practice, either function works as long as you apply the correct phase shift. Most PreCal 12 questions will tell you whether to use sine or cosine, or the starting position will make the choice obvious.

Part 2: Right Triangle Applications

Sometimes trigonometry is used to find unknown distances or angles in real-world settings using right triangles. The two most common situations are **angles of elevation** and **angles of depression**.

- **Angle of elevation:** The angle measured *upward* from the horizontal to a point above you. (You are looking up.)
- **Angle of depression:** The angle measured *downward* from the horizontal to a point below you. (You are looking down from a height.)

In both cases, you draw a right triangle and apply SOH-CAH-TOA:

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

When you know the angle and one side, you can solve for the missing side. When you know two sides, you can solve for the angle using an inverse trig function.

Pro Tips

- **Find d and a first.** Once you have the maximum and minimum values, you can immediately calculate the midline and amplitude. This locks in two of the four constants before you even think about b or c.
- **Use $-\cos$ for "starts at the bottom."** If a situation begins at its lowest point (like a Ferris wheel rider starting at ground level), a negative cosine requires no phase shift. This keeps the equation clean.
- **Always check your equation with a known point.** After building your model, substitute a known time value back in. If you get the right output, your equation is correct.
- **For angle problems, draw a diagram first.** Sketch the triangle, label the sides, and mark the angle. This almost always makes it obvious which trig ratio to use.
- **Angle of elevation = angle of depression.** When you have a horizontal line between two points at different heights, the angle of elevation from the lower point equals the angle of depression from the upper point. (Alternate interior angles between parallel lines.)

Examples

Example 1. *Ferris Wheel*

A Ferris wheel has a radius of 15 m, and its centre is 18 m above the ground. It completes one full revolution every 40 seconds. A rider starts at the *bottom* of the wheel. Write an equation for the rider's height h (in metres) as a function of time t (in seconds).

Step 1: Identify the maximum and minimum heights.

The centre of the wheel is 18 m high. The wheel has a radius of 15 m, so:

$$\text{Maximum height} = 18 + 15 = 33 \text{ m} \qquad \text{Minimum height} = 18 - 15 = 3 \text{ m}$$

Step 2: Find a and d.

$$a = \frac{33-3}{2} = \frac{30}{2} = 15 \qquad\qquad d = \frac{33+3}{2} = \frac{36}{2} = 18$$

Step 3: Find b using the period.

The period is 40 seconds:

$$b = \frac{2\pi}{40} = \frac{\pi}{20}$$

Step 4: Choose the function type.

The rider starts at the *minimum* height (bottom of the wheel) at $t = 0$. A negative cosine starts at its minimum with no phase shift needed. So we use $-\cos$ and set $c = 0$:

$$h(t) = -15\cos\left(\frac{\pi}{20}t\right) + 18$$

Step 5: Verify with known values.

At $t = 0$ (start, at the bottom):

$$h(0) = -15\cos(0) + 18 = -15(1) + 18 = 3 \text{ m} \quad \checkmark$$

At $t = 20$ (halfway through, at the top):

$$h(20) = -15\cos\left(\frac{\pi}{20}\cdot 20\right) + 18 = -15\cos(\pi) + 18 = -15(-1) + 18 = 33 \text{ m} \quad \checkmark$$

$$\boxed{h(t) = -15\cos\left(\frac{\pi}{20}t\right) + 18}$$

Example 2. *Ocean Tides*

At a harbour, the water depth follows a sinusoidal pattern. The high tide is 9 m at $t = 2$ hours, and the low tide is 1 m. The tidal cycle repeats every 12 hours. Write a cosine equation for the water depth h (in metres) as a function of time t (in hours).

Step 1: Find a and d.

$$a = \frac{9-1}{2} = \frac{8}{2} = 4 \qquad\qquad d = \frac{9+1}{2} = \frac{10}{2} = 5$$

Step 2: Find b.

$$b = \frac{2\pi}{12} = \frac{\pi}{6}$$

Step 3: Find the phase shift c.

The high tide (maximum) occurs at $t = 2$ hours. Since we are using cosine, which starts at its maximum, the phase shift is:

$$c = 2$$

Step 4: Write the equation.

$$h(t) = 4\cos\left(\frac{\pi}{6}(t-2)\right) + 5$$

Step 5: Verify.

At $t = 2$ (high tide):

$$h(2) = 4\cos\left(\frac{\pi}{6}(0)\right) + 5 = 4(1) + 5 = 9 \text{ m} \quad \checkmark$$

At $t = 8$ (low tide, 6 hours after the high):

$$h(8) = 4\cos\left(\frac{\pi}{6}(6)\right) + 5 = 4\cos(\pi) + 5 = 4(-1) + 5 = 1 \text{ m} \quad \checkmark$$

$$\boxed{h(t) = 4\cos\left(\frac{\pi}{6}(t-2)\right) + 5}$$

Example 3. *Angle of Elevation*

A person stands 50 m away from the base of a building. The angle of elevation to the top of the building is $32°$. Find the height of the building. Round your answer to two decimal places.

Step 1: Draw the right triangle and label the sides.

The person is on flat ground, so the horizontal distance (adjacent side) is 50 m. The height of the building is the opposite side. The angle of elevation is $32°$.

Step 2: Choose the correct trig ratio.

We know the adjacent side and want the opposite side. This is the tangent ratio:

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

Step 3: Substitute and solve.

$$\tan(32°) = \frac{h}{50}$$

$$h = 50 \times \tan(32°)$$

$$h \approx 50 \times 0.6249$$

$$h \approx 31.24 \text{ m}$$

$$\boxed{h \approx 31.24 \text{ m}}$$

Common Mistakes

- **Confusing amplitude with maximum value.** The amplitude a is the distance from the midline to the maximum — not the maximum itself. Always use $a = \dfrac{\max - \min}{2}$, not just the maximum value.
- **Forgetting the negative sign for "starts at the bottom."** When a situation starts at its minimum (like a Ferris wheel at the bottom), you need $-\cos$, not $+\cos$. Using $+\cos$ would give a graph that starts at the top instead.
- **Getting the period formula backwards.** Remember: $b = \dfrac{2\pi}{\text{Period}}$. Many students accidentally write $b = \text{Period}$, which gives a completely wrong graph.
- **Mixing up phase shift direction.** The form $b(x - c)$ shifts the graph *right* when c is positive. If you write $b(x + c)$, the shift is to the *left*. Always rewrite in the form $(x - c)$ and identify c carefully.
- **Using sine instead of cosine (or vice versa) without adjusting the phase shift.** If the problem starts at the maximum and you choose sine, you must include a phase shift of $-\dfrac{\pi}{2b}$ to move it to the right position. It is usually cleaner to choose the function that matches the starting point naturally.

- **In angle problems: using the wrong trig ratio.** Always draw the triangle and label which sides are opposite, adjacent, and hypotenuse *relative to the given angle* before picking a ratio.

Worksheet

1. For the function $y = 3\sin\big(2(x - \frac{\pi}{4})\big) + 1$, identify:

(a) the amplitude

(b) the period

(c) the phase shift

(d) the equation of the midline

2. A sinusoidal function has a maximum value of 6, a minimum value of -2, and a period of 4π. The graph starts at its maximum at $x = 0$. Write an equation in the form $y = a\cos(bx) + d$.

3. A Ferris wheel has a radius of 12 m, and its centre is 15 m above the ground. It takes 30 seconds to complete one full revolution. A rider boards at the **bottom** of the wheel.

(a) Write an equation for the rider's height $h(t)$ in metres.

(b) Find the rider's height at $t = 10$ seconds.

4. A surveyor stands 40 m from the base of a vertical cliff. The angle of elevation to the top of the cliff is 28°. Find the height of the cliff. Round to two decimal places.

5. Using the tide equation from Example 2, $h(t) = 4\cos\left(\frac{\pi}{6}(t-2)\right) + 5$, find all times t in the interval $[0, 12]$ when the water depth is exactly 7 m. Show all algebraic steps.

6. **(Find the angle)** A flagpole casts a 45 m horizontal shadow. The flagpole is 80 m tall. Find the angle of elevation of the sun. Round to two decimal places.

Answer Key

1. $y = 3\sin\big(2(x - \frac{\pi}{4})\big) + 1$

Step 1: Match to the standard form $y = a\sin(b(x-c)) + d$.
Reading off the constants directly:

$$a = 3, \qquad b = 2, \qquad c = \frac{\pi}{4}, \qquad d = 1$$

Step 2: Calculate each feature.
(a) Amplitude:

$$|a| = |3| = 3$$

(b) Period:

$$\text{Period} = \frac{2\pi}{b} = \frac{2\pi}{2} = \pi$$

(c) Phase shift:
The value $c = \dfrac{\pi}{4}$ and it appears as $(x - \dfrac{\pi}{4})$, so the graph shifts to the **right** by $\dfrac{\pi}{4}$.
(d) Midline:

$$y = d = 1$$

$$\boxed{\text{Amplitude} = 3, \quad \text{Period} = \pi, \quad \text{Phase shift} = \frac{\pi}{4} \text{ right}, \quad \text{Midline: } y = 1}$$

2. Maximum $= 6$, minimum $= -2$, period $= 4\pi$, starts at maximum.

Step 1: Find the amplitude a.

$$a = \frac{\max - \min}{2} = \frac{6-(-2)}{2} = \frac{8}{2} = 4$$

Step 2: Find the vertical shift d.

$$d = \frac{\max + \min}{2} = \frac{6+(-2)}{2} = \frac{4}{2} = 2$$

Step 3: Find b.

$$b = \frac{2\pi}{\text{Period}} = \frac{2\pi}{4\pi} = \frac{1}{2}$$

Step 4: Choose the function and check for a phase shift.

The graph starts at its maximum when $x = 0$. Cosine also starts at its maximum (when there is no phase shift), so we use cos with $c = 0$.

The equation is:

$$y = 4\cos\left(\frac{1}{2}x\right) + 2$$

Step 5: Verify key points.

At $x = 0$ (maximum):

$$y = 4\cos(0) + 2 = 4(1) + 2 = 6 \quad \checkmark$$

At $x = 2\pi$ (minimum, after half a period):

$$y = 4\cos\left(\frac{1}{2}\cdot 2\pi\right) + 2 = 4\cos(\pi) + 2 = 4(-1) + 2 = -2 \quad \checkmark$$

At $x = 4\pi$ (back to maximum, after one full period):

$$y = 4\cos\left(\frac{1}{2}\cdot 4\pi\right) + 2 = 4\cos(2\pi) + 2 = 4(1) + 2 = 6 \quad \checkmark$$

$$\boxed{y = 4\cos\left(\frac{1}{2}x\right) + 2}$$

3. Ferris wheel: radius = 12 m, centre = 15 m high, period = 30 s, starts at bottom.

(a) Write the equation for $h(t)$.

Step 1: Identify the maximum and minimum heights.

$$\text{Maximum} = 15 + 12 = 27 \text{ m} \qquad \text{Minimum} = 15 - 12 = 3 \text{ m}$$

Step 2: Find a and d.

$$a = \frac{27-3}{2} = \frac{24}{2} = 12 \qquad d = \frac{27+3}{2} = \frac{30}{2} = 15$$

Step 3: Find b.

$$b = \frac{2\pi}{30} = \frac{\pi}{15}$$

Step 4: Choose the function.

The rider starts at the minimum (bottom) at $t = 0$. A negative cosine starts at its minimum with no phase shift needed:

$$h(t) = -12\cos\left(\frac{\pi}{15}t\right) + 15$$

Step 5: Verify.

At $t = 0$ (bottom):

$$h(0) = -12\cos(0) + 15 = -12(1) + 15 = 3 \text{ m} \quad \checkmark$$

At $t = 15$ (top, halfway through):

$$h(15) = -12\cos(\pi) + 15 = -12(-1) + 15 = 27 \text{ m} \quad \checkmark$$

$$\boxed{h(t) = -12\cos\left(\frac{\pi}{15}t\right) + 15}$$

(b) Find the height at $t = 10$ seconds.

Step 1: Substitute $t = 10$ into the equation.

$$h(10) = -12\cos\left(\frac{\pi}{15} \cdot 10\right) + 15$$

Step 2: Simplify the angle inside the cosine.

$$\frac{\pi}{15} \cdot 10 = \frac{10\pi}{15} = \frac{2\pi}{3}$$

Step 3: Evaluate $\cos\left(\frac{2\pi}{3}\right)$.

The angle $\frac{2\pi}{3}$ is in Quadrant II. Its reference angle is $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$. Cosine is negative in Quadrant II:

$$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

Step 4: Substitute and solve.

$$h(10) = -12 \times \left(-\frac{1}{2}\right) + 15 = 6 + 15 = 21 \text{ m}$$

$$\boxed{h(10) = 21 \text{ m}}$$

4. Surveyor stands 40 m from a cliff, angle of elevation = 28°.

Step 1: Set up the right triangle.

The horizontal distance (adjacent to the angle) is 40 m. The height of the cliff is the side opposite the angle.

Step 2: Choose the trig ratio.

We know the adjacent side and want the opposite side, so we use tangent:

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

Step 3: Substitute.

$$\tan(28^\circ) = \frac{h}{40}$$

Step 4: Solve for h.

$$h = 40 \times \tan(28^\circ)$$

$$h \approx 40 \times 0.5317$$

$$h \approx 21.27 \text{ m}$$

$$\boxed{h \approx 21.27 \text{ m}}$$

5. Find all $t \in [0, 12]$ where $h(t) = 7$ using $h(t) = 4\cos\left(\frac{\pi}{6}(t-2)\right) + 5$.

Step 1: Set the equation equal to 7.

$$4\cos\left(\frac{\pi}{6}(t-2)\right) + 5 = 7$$

Step 2: Isolate the cosine.

$$4\cos\left(\frac{\pi}{6}(t-2)\right) = 2$$

$$\cos\left(\frac{\pi}{6}(t-2)\right) = \frac{1}{2}$$

Step 3: Solve for the angle inside the cosine.

We need an angle ϕ such that $\cos(\phi) = \frac{1}{2}$.

From our special triangles:

$$\phi = \frac{\pi}{3} \qquad \text{or} \qquad \phi = -\frac{\pi}{3}$$

(Cosine equals $\frac{1}{2}$ in Quadrant I and Quadrant IV.)

So:

$$\frac{\pi}{6}(t-2) = \frac{\pi}{3} \qquad \text{or} \qquad \frac{\pi}{6}(t-2) = -\frac{\pi}{3}$$

Step 4: Solve each equation for t.

Case 1:

$$\frac{\pi}{6}(t-2) = \frac{\pi}{3}$$

$$t - 2 = \frac{\pi}{3} \div \frac{\pi}{6} = \frac{\pi}{3} \times \frac{6}{\pi} = 2$$

$$t = 4$$

Case 2:

$$\frac{\pi}{6}(t-2) = -\frac{\pi}{3}$$

$$t - 2 = -\frac{\pi}{3} \times \frac{6}{\pi} = -2$$

$$t = 0$$

Step 5: Check that both solutions are in $[0, 12]$.

$t = 0$: in $[0, 12]$. ✓ $t = 4$: in $[0, 12]$. ✓

Step 6: Verify both solutions.

At $t = 0$:

$$h(0) = 4\cos\left(\frac{\pi}{6}(0-2)\right) + 5 = 4\cos\left(-\frac{\pi}{3}\right) + 5 = 4 \times \frac{1}{2} + 5 = 7 \quad \checkmark$$

At $t = 4$:

$$h(4) = 4\cos\left(\frac{\pi}{6}(4-2)\right) + 5 = 4\cos\left(\frac{\pi}{3}\right) + 5 = 4 \times \frac{1}{2} + 5 = 7 \quad \checkmark$$

$$\boxed{t = 0 \text{ hours and } t = 4 \text{ hours}}$$

6. Shadow = 45 m, flagpole = 80 m. Find the angle of elevation of the sun.

Step 1: Set up the right triangle.
The flagpole is the side opposite the angle of elevation. The shadow is the adjacent side.
Step 2: Choose the trig ratio.
We know both the opposite and adjacent sides, so we use tangent:

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{80}{45}$$

Step 3: Solve for θ using the inverse tangent.

$$\theta = \arctan\left(\frac{80}{45}\right)$$

Step 4: Simplify and calculate.

$$\frac{80}{45} = \frac{16}{9} \approx 1.7\overline{7}$$

$$\theta = \arctan(1.7\overline{7}) \approx 60.64^\circ$$

$$\boxed{\theta \approx 60.64^\circ}$$

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