Lesson 7: Graphing Inverse Trigonometric Functions
ApexMath

Notes

What Is an Inverse Function?

Before diving into inverse *trig* functions, let's make sure the idea of an inverse function is clear.

A regular function takes an input and gives you an output. An **inverse function** does the opposite — it takes the *output* and gives you back the *input*.

For example, if $f(x) = x^2$ outputs 9 when you put in 3, then the inverse function would take 9 and give you back 3.

We write the inverse of f as f^{-1}.

Why We Need Restricted Domains

Here is the key problem with trig functions: they repeat. Sine, cosine, and tangent all go through the same output values over and over as the angle increases.

For a function to have an inverse, each output must come from *exactly one* input. Because trig functions repeat, we have to **restrict the domain** to a small window where the function never repeats an output.

Think of it this way: if $\sin(\theta) = \frac{1}{2}$, the angle could be $\frac{\pi}{6}$, or $\frac{5\pi}{6}$, or infinitely many others. To make a proper inverse, we pick just *one* agreed-upon interval and stick to it.

The Three Inverse Trig Functions

1. Arcsine — written $\arcsin(x)$ or $\sin^{-1}(x)$

- Asks the question: "What angle has a sine value of x?"
- **Domain:** $-1 \leq x \leq 1$
- **Range:** $-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$ (answers are always in Quadrant I or IV, or on the axes)

$$y = \arcsin(x) \quad \Longleftrightarrow \quad \sin(y) = x, \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

2. Arccosine — written $\arccos(x)$ or $\cos^{-1}(x)$

- Asks the question: "What angle has a cosine value of x?"
- **Domain:** $-1 \leq x \leq 1$
- **Range:** $0 \leq y \leq \pi$ (answers are always in Quadrant I or II, or on the axes)

$$y = \arccos(x) \quad \Longleftrightarrow \quad \cos(y) = x, \quad 0 \leq y \leq \pi$$

3. Arctangent — written $\arctan(x)$ or $\tan^{-1}(x)$

- Asks the question: "What angle has a tangent value of x?"
- **Domain:** All real numbers ($-\infty < x < \infty$)
- **Range:** $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$ (open interval — the function never actually reaches $\pm\frac{\pi}{2}$; those are *horizontal asymptotes*)

$$y = \arctan(x) \quad \Longleftrightarrow \quad \tan(y) = x, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

A Quick Reference Table

Function	**Domain**	**Range**
$y = \arcsin(x)$	$[-1,\ 1]$	$\left[-\dfrac{\pi}{2},\ \dfrac{\pi}{2}\right]$
$y = \arccos(x)$	$[-1,\ 1]$	$[0,\ \pi]$
$y = \arctan(x)$	$(-\infty,\ \infty)$	$\left(-\dfrac{\pi}{2},\ \dfrac{\pi}{2}\right)$

Graph Shapes to Know

- $y = \arcsin(x)$: S-shaped curve passing through $(-1, -\frac{\pi}{2})$, $(0, 0)$, and $(1, \frac{\pi}{2})$. It is an *increasing* function.
- $y = \arccos(x)$: Decreasing curve passing through $(-1, \pi)$, $(0, \frac{\pi}{2})$, and $(1, 0)$.
- $y = \arctan(x)$: S-shaped curve that flattens out as $x \to \pm\infty$. It has horizontal asymptotes at $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$. Passes through the origin $(0, 0)$.

Pro Tip

- **Rewrite the question.** Whenever you see $\arcsin(x)$, mentally replace it with: *"The angle whose sine is x."* This rewording almost always makes the problem easier to set up.
- **Know your special triangles.** Most exam questions use values from the 30-60-90 or 45-45-90 triangles. If you know that $\sin(30°) = \frac{1}{2}$, then you immediately know $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$.
- **Always check the range.** The range tells you which quadrant the answer must land in. If you get an angle outside the stated range, it is wrong — even if the trig value is correct.
- **$\sin^{-1}$ does NOT mean $\frac{1}{\sin}$.** The superscript -1 here means *inverse function*, not a reciprocal. $\dfrac{1}{\sin(x)}$ is $\csc(x)$, which is a completely different thing.

Examples

Example 1. Evaluate $\arcsin\left(\dfrac{1}{2}\right)$.

Step 1: Rewrite the question.

We are looking for an angle y such that $\sin(y) = \dfrac{1}{2}$. We also need y to be inside the range of arcsine: $-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$.

Step 2: Think about your special triangles.

From the 30-60-90 triangle, we know:

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

Step 3: Check the range.

Is $\dfrac{\pi}{6}$ inside $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$?

$$-\frac{\pi}{2} \leq \frac{\pi}{6} \leq \frac{\pi}{2} \quad \checkmark$$

Step 4: State the answer.

$$\boxed{\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}}$$

Example 2. Evaluate $\arccos(-1)$.

Step 1: Rewrite the question.
We need an angle y such that $\cos(y) = -1$. The range of arccosine is $0 \le y \le \pi$.

Step 2: Identify where cosine equals -1.
Cosine equals -1 at the angle π (that is, 180°).

$$\cos(\pi) = -1$$

Step 3: Check the range.
Is π inside $[0, \pi]$?

$$0 \le \pi \le \pi \quad \checkmark$$

Step 4: State the answer.

$$\boxed{\arccos(-1) = \pi}$$

Example 3. Evaluate $\arctan\left(\sqrt{3}\right)$.

Step 1: Rewrite the question.
We need an angle y such that $\tan(y) = \sqrt{3}$. The range of arctangent is $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Step 2: Think about your special triangles.
From the 30-60-90 triangle:

$$\tan\left(\frac{\pi}{3}\right) = \frac{\sin(\pi/3)}{\cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

Step 3: Check the range.
Is $\frac{\pi}{3}$ inside $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$?

$$-\frac{\pi}{2} < \frac{\pi}{3} < \frac{\pi}{2} \quad \checkmark$$

Step 4: State the answer.

$$\boxed{\arctan\left(\sqrt{3}\right) = \frac{\pi}{3}}$$

Example 4. Evaluate $\arcsin\left(\sin\left(\frac{5\pi}{6}\right)\right)$.

This is a *composition* problem. Many students assume the answer is just $\frac{5\pi}{6}$ because they think the inverse "undoes" the function. That is only true if the angle is already inside the range of arcsine. Let us check carefully.

Step 1: Evaluate the inside first.

Compute $\sin\left(\frac{5\pi}{6}\right)$.

The angle $\frac{5\pi}{6}$ is in Quadrant II. Its reference angle is $\pi - \frac{5\pi}{6} = \frac{\pi}{6}$. Sine is positive in Quadrant II, so:

$$\sin\left(\frac{5\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

Step 2: Now evaluate the outside.

We now need:

$$\arcsin\left(\frac{1}{2}\right)$$

From Example 1, we already know this equals $\frac{\pi}{6}$.

Step 3: Confirm why the answer is NOT $\frac{5\pi}{6}$.

The range of arcsine is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The angle $\frac{5\pi}{6}$ is *outside* this range (it is larger than $\frac{\pi}{2}$), so arcsine cannot return $\frac{5\pi}{6}$. Arcsine found the equivalent angle inside the allowed range.

Step 4: State the answer.

$$\boxed{\arcsin\left(\sin\left(\frac{5\pi}{6}\right)\right) = \frac{\pi}{6}}$$

Common Mistakes

- **Ignoring the range.** The most common error is forgetting that arcsin can only return values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. An answer like $\frac{5\pi}{6}$ or $\frac{3\pi}{4}$ is always wrong for arcsine.

- **Thinking $\sin^{-1}(x) = \frac{1}{\sin(x)}$.** It does not. The -1 exponent on a function name means *inverse*, not reciprocal.

- **Cancelling too quickly in compositions.** $\arcsin(\sin(\theta)) = \theta$ is only true when $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. If the angle is outside that interval, you must find the equivalent angle that *is* inside the range.

- **Using degrees when the problem is in radians (or vice versa).** Make sure you know which unit the problem uses and stay consistent.

- **Mixing up arcsine and arccosine ranges.** Arcsine gives angles in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (Quadrants I and IV). Arccosine gives angles in $[0, \pi]$ (Quadrants I and II). Confusing these leads to wrong quadrant errors.

Worksheet

Evaluate each expression. Give your answers in radians.

1. $\arcsin\left(\frac{\sqrt{3}}{2}\right)$

2. $\arccos(0)$

3. $\arctan(\sqrt{3})$

4. $\arcsin\left(-\frac{1}{2}\right)$

5. $\arccos(-1)$

6. $\arctan(-1)$

7. $\arcsin\left(\frac{\sqrt{2}}{2}\right)$

8. $\arccos\left(\frac{1}{2}\right)$

9. $\arctan\left(\frac{1}{\sqrt{3}}\right)$

10. $\arcsin(1)$

11. (Composition) $\cos\left(\arcsin\left(\frac{1}{2}\right)\right)$

Hint: First find the angle, then evaluate cosine at that angle.

12. (Composition) $\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right)$

Hint: Check whether the angle is inside the range of arccosine before cancelling.

Answer Key

1. $\arcsin\left(\dfrac{\sqrt{3}}{2}\right)$

Step 1: We need an angle y with $\sin(y) = \dfrac{\sqrt{3}}{2}$, where $y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.
Step 2: From the 30-60-90 triangle:

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

Step 3: Check the range: $\dfrac{\pi}{3} \approx 1.047$ is between $-\dfrac{\pi}{2} \approx -1.571$ and $\dfrac{\pi}{2} \approx 1.571$. ✓

$$\boxed{\frac{\pi}{3}}$$

2. $\arccos(0)$

Step 1: We need an angle y with $\cos(y) = 0$, where $y \in [0, \pi]$.
Step 2: Cosine equals zero at $y = \dfrac{\pi}{2}$ (on the unit circle, this is the top of the circle where the x-coordinate is 0).

$$\cos\left(\frac{\pi}{2}\right) = 0$$

Step 3: Check the range: $\dfrac{\pi}{2}$ is in $[0, \pi]$. ✓

$$\boxed{\frac{\pi}{2}}$$

3. $\arctan(\sqrt{3})$

Step 1: We need an angle y with $\tan(y) = \sqrt{3}$, where $y \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.
Step 2: Compute:

$$\tan\left(\frac{\pi}{3}\right) = \frac{\sin(\pi/3)}{\cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

Step 3: Check the range: $\dfrac{\pi}{3}$ is inside $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. ✓

$$\boxed{\frac{\pi}{3}}$$

4. $\arcsin\left(-\frac{1}{2}\right)$

Step 1: We need y with $\sin(y) = -\frac{1}{2}$, where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Step 2: We know $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. Since the value is *negative*, the angle must be in Quadrant IV, which means we use the negative:

$$\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$$

Step 3: Check the range: $-\frac{\pi}{6}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. ✓

$$\boxed{-\frac{\pi}{6}}$$

5. $\arccos(-1)$

Step 1: We need y with $\cos(y) = -1$, where $y \in [0, \pi]$.

Step 2: On the unit circle, cosine equals -1 at the leftmost point, which corresponds to the angle π:

$$\cos(\pi) = -1$$

Step 3: Check the range: π is in $[0, \pi]$. ✓

$$\boxed{\pi}$$

6. $\arctan(-1)$

Step 1: We need y with $\tan(y) = -1$, where $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Step 2: We know $\tan\left(\frac{\pi}{4}\right) = 1$. Since the value is *negative*, the angle must be in Quadrant IV:

$$\tan\left(-\frac{\pi}{4}\right) = -1$$

Step 3: Check the range: $-\frac{\pi}{4}$ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. ✓

$$\boxed{-\frac{\pi}{4}}$$

7. $\arcsin\left(\dfrac{\sqrt{2}}{2}\right)$

Step 1: We need y with $\sin(y) = \dfrac{\sqrt{2}}{2}$, where $y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.
Step 2: From the 45-45-90 triangle:

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

Step 3: Check the range: $\dfrac{\pi}{4}$ is in $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$. ✓

$$\boxed{\frac{\pi}{4}}$$

8. $\arccos\left(\dfrac{1}{2}\right)$

Step 1: We need y with $\cos(y) = \dfrac{1}{2}$, where $y \in [0, \pi]$.
Step 2: From the 30-60-90 triangle:

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

Step 3: Check the range: $\dfrac{\pi}{3}$ is in $[0, \pi]$. ✓

$$\boxed{\frac{\pi}{3}}$$

9. $\arctan\left(\dfrac{1}{\sqrt{3}}\right)$

Step 1: We need y with $\tan(y) = \dfrac{1}{\sqrt{3}}$, where $y \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.
Step 2: From the 30-60-90 triangle:

$$\tan\left(\frac{\pi}{6}\right) = \frac{\sin(\pi/6)}{\cos(\pi/6)} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

Step 3: Check the range: $\dfrac{\pi}{6}$ is in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. ✓

$$\boxed{\frac{\pi}{6}}$$

10. $\arcsin(1)$

Step 1: We need y with $\sin(y) = 1$, where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Step 2: On the unit circle, sine equals 1 at the top of the circle, which is the angle $\frac{\pi}{2}$:

$$\sin\left(\frac{\pi}{2}\right) = 1$$

Step 3: This is actually the boundary of the range, so it is allowed. ✓

$$\boxed{\frac{\pi}{2}}$$

11. $\cos\left(\arcsin\left(\frac{1}{2}\right)\right)$

Step 1: Work from the inside out. First evaluate $\arcsin\left(\frac{1}{2}\right)$.

From Question 1's structure:

$$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Step 2: Now the expression becomes:

$$\cos\left(\frac{\pi}{6}\right)$$

Step 3: From the 30-60-90 triangle:

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$\boxed{\cos\left(\arcsin\left(\frac{1}{2}\right)\right) = \frac{\sqrt{3}}{2}}$$

12. $\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right)$

Step 1: Work from the inside out. Evaluate $\cos\left(\frac{2\pi}{3}\right)$.

The angle $\frac{2\pi}{3}$ is in Quadrant II. Its reference angle is $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$. Cosine is *negative* in Quadrant II:

$$\cos\left(\frac{2\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$$

Step 2: Now the expression becomes:

$$\arccos\left(-\frac{1}{2}\right)$$

Step 3: We need y with $\cos(y) = -\frac{1}{2}$, where $y \in [0, \pi]$.

Since the value is negative, the angle must be in Quadrant II. The reference angle for $\cos = \frac{1}{2}$ is $\frac{\pi}{3}$. The Quadrant II angle with that reference is $\pi - \frac{\pi}{3}$:

$$\cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

Step 4: Check the range: $\frac{2\pi}{3}$ is in $[0, \pi]$. ✓

Note: This time the composition *did* return the original angle because $\frac{2\pi}{3}$ was already inside the range of arccosine $[0, \pi]$.

$$\boxed{\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}}$$

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