Lesson 9: Solving Trigonometric Equations Through Graphing and Algebra
ApexMath

Notes

A trigonometric equation is any equation that contains a trig function such as sin, cos, or tan. Solving one means finding the angle values that make the equation true.

There are two main approaches: **graphing** and **algebra**. In this lesson you will learn both and understand when each one is most useful.

Part 1: Solving by Graphing

The graphing method is a visual approach. Here is how it works:

Step 1. Rewrite the equation so one side is a trig function and the other side is a constant. For example, $\sin(x) = \frac{1}{2}$.

Step 2. Graph the trig function (e.g., $y = \sin x$) on the interval you care about.

Step 3. On the same set of axes, graph the horizontal line $y = \frac{1}{2}$.

Step 4. The x-coordinates of the intersection points are your solutions.

This method is useful for getting a visual sense of how many solutions exist and roughly where they are. It is less precise than the algebraic method unless you use technology.

Part 2: Solving Algebraically

The algebraic method gives exact answers. The core idea is:

Step 1. Isolate the trig function on one side of the equation.

Step 2. Use your knowledge of special triangles and the unit circle to find the reference angle.

Step 3. Determine which quadrants give the correct sign for the trig value.

Step 4. Write all angles in the given interval that satisfy the equation.

Finding Solutions Using the CAST Rule

The CAST rule tells you the sign of each trig function in each quadrant.

Quadrant I	**Quadrant II**	**Quadrant III**	**Quadrant IV**
All positive	**Sin** positive	**Tan** positive	**Cos** positive

Once you know the reference angle and which quadrants apply, you can build each solution:

- **Quadrant I:** $\theta = \theta_{\text{ref}}$
- **Quadrant II:** $\theta = \pi - \theta_{\text{ref}}$
- **Quadrant III:** $\theta = \pi + \theta_{\text{ref}}$
- **Quadrant IV:** $\theta = 2\pi - \theta_{\text{ref}}$

Part 3: General Solutions

When no interval is given, you must express the **general solution** — all possible angles, not just those in $[0, 2\pi]$.

Since trig functions repeat every 2π (for sine and cosine) or every π (for tangent), general solutions add multiples of the period:

- **Sine and cosine equations** have two families of solutions. Each family adds $2\pi n$:

$$x = \theta_1 + 2\pi n \quad \text{OR} \quad x = \theta_2 + 2\pi n, \qquad n \in \mathbb{Z}$$

- **Tangent equations** have one family of solutions. Each family adds πn:

$$x = \theta_{\text{ref}} + \pi n, \qquad n \in \mathbb{Z}$$

The symbol $n \in \mathbb{Z}$ means n can be any integer: $\ldots, -2, -1, 0, 1, 2, \ldots$

Part 4: Equations with a Coefficient on x

When the argument of the trig function is bx instead of just x — for example, $\sin(2x) = \frac{1}{2}$ — you must expand the solution interval before dividing.

Step 1. Let $u = bx$. Determine the new interval for u. If $x \in [0, 2\pi]$, then $u = 2x \in [0, 4\pi]$.

Step 2. Solve $\sin(u) = \frac{1}{2}$ on the expanded interval $[0, 4\pi]$. Find all values of u.

Step 3. Divide every u solution by b to get the x solutions.

Part 5: Quadratic Trig Equations

Some equations are quadratic in a trig function. For example:

$$2\sin^2(x) - \sin(x) - 1 = 0$$

To solve these, treat the trig function like a variable. Let $u = \sin(x)$ and factor or use the quadratic formula on u. Then solve for x from each value of u.

Pro Tips

- **Always find the reference angle first.** No matter which quadrant the answer is in, the reference angle is the starting point. It is always positive and always between 0 and $\frac{\pi}{2}$.
- **Check the sign before writing your answer.** After finding the reference angle, use CAST to confirm which quadrants give the correct sign. A positive sine value means Quadrants I and II. A negative cosine value means Quadrants II and III.
- **For $\sin(bx)$ problems, expand the interval first.** This is the step most students forget. If $x \in [0, 2\pi]$ and the equation has $\sin(2x)$, you must find all solutions in $[0, 4\pi]$ for the inner expression, then divide by 2 at the end.
- **For quadratic equations, factor first.** Try to factor before reaching for the quadratic formula. Most exam problems are designed to factor cleanly.
- **Always verify your solutions.** Substitute each answer back into the original equation. Extraneous solutions can appear, especially when squaring both sides.

Examples

Example 1. Solve $\sin(x) = \dfrac{1}{2}$ on $[0, 2\pi]$.

Step 1: Find the reference angle.

We need an angle whose sine equals $\dfrac{1}{2}$. From the 30-60-90 triangle:

$$\theta_{\text{ref}} = \frac{\pi}{6}$$

Step 2: Determine which quadrants apply.

Sine is positive, so we need Quadrants I and II (where sine is positive, according to CAST).

Step 3: Write the solutions in each quadrant.

Quadrant I:

$$x = \theta_{\text{ref}} = \frac{\pi}{6}$$

Quadrant II:

$$x = \pi - \theta_{\text{ref}} = \pi - \frac{\pi}{6} = \frac{6\pi}{6} - \frac{\pi}{6} = \frac{5\pi}{6}$$

Step 4: Verify both solutions.

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \quad \checkmark \qquad \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2} \quad \checkmark$$

$$\boxed{x = \frac{\pi}{6} \quad \text{and} \quad x = \frac{5\pi}{6}}$$

Example 2. Solve $2\cos(x) - 1 = 0$ on $[0, 2\pi]$.

Step 1: Isolate the trig function.

$$2\cos(x) - 1 = 0$$

$$2\cos(x) = 1$$

$$\cos(x) = \frac{1}{2}$$

Step 2: Find the reference angle.

We need an angle whose cosine equals $\frac{1}{2}$. From the 30-60-90 triangle:

$$\theta_{\text{ref}} = \frac{\pi}{3}$$

Step 3: Determine which quadrants apply.

Cosine is positive, so we need Quadrants I and IV.

Step 4: Write the solutions.

Quadrant I:

$$x = \frac{\pi}{3}$$

Quadrant IV:

$$x = 2\pi - \frac{\pi}{3} = \frac{6\pi}{3} - \frac{\pi}{3} = \frac{5\pi}{3}$$

Step 5: Verify.

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \quad \checkmark \qquad \cos\left(\frac{5\pi}{3}\right) = \frac{1}{2} \quad \checkmark$$

$$\boxed{x = \frac{\pi}{3} \quad \text{and} \quad x = \frac{5\pi}{3}}$$

Example 3. Solve $2\sin^2(x) - \sin(x) - 1 = 0$ on $[0, 2\pi]$.

Step 1: Recognise the quadratic structure.
Let $u = \sin(x)$. The equation becomes:

$$2u^2 - u - 1 = 0$$

Step 2: Factor.

$$(2u+1)(u-1) = 0$$

Step 3: Solve for u.

$$2u + 1 = 0 \quad \Rightarrow \quad u = -\frac{1}{2} \qquad \text{or} \qquad u - 1 = 0 \quad \Rightarrow \quad u = 1$$

So $\sin(x) = -\dfrac{1}{2}$ or $\sin(x) = 1$.
Step 4: Solve $\sin(x) = 1$.
Sine equals 1 only at the top of the unit circle:

$$x = \frac{\pi}{2}$$

Step 5: Solve $\sin(x) = -\dfrac{1}{2}$.
The reference angle for $\sin = \dfrac{1}{2}$ is $\dfrac{\pi}{6}$. Sine is negative in Quadrants III and IV:
Quadrant III:

$$x = \pi + \frac{\pi}{6} = \frac{6\pi}{6} + \frac{\pi}{6} = \frac{7\pi}{6}$$

Quadrant IV:

$$x = 2\pi - \frac{\pi}{6} = \frac{12\pi}{6} - \frac{\pi}{6} = \frac{11\pi}{6}$$

Step 6: Verify all three solutions.

$$2\sin^2\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) - 1 = 2(1)^2 - 1 - 1 = 0 \quad \checkmark$$

$$2\sin^2\left(\frac{7\pi}{6}\right) - \sin\left(\frac{7\pi}{6}\right) - 1 = 2\left(\tfrac{1}{4}\right) - \left(-\tfrac{1}{2}\right) - 1 = \tfrac{1}{2} + \tfrac{1}{2} - 1 = 0 \quad \checkmark$$

$$2\sin^2\left(\frac{11\pi}{6}\right) - \sin\left(\frac{11\pi}{6}\right) - 1 = 2\left(\tfrac{1}{4}\right) - \left(-\tfrac{1}{2}\right) - 1 = \tfrac{1}{2} + \tfrac{1}{2} - 1 = 0 \quad \checkmark$$

$$\boxed{x = \frac{\pi}{2}, \quad x = \frac{7\pi}{6}, \quad x = \frac{11\pi}{6}}$$

Example 4. Find the general solution for $\tan(x) = -1$.

Step 1: Find the reference angle.

We need an angle whose tangent equals 1 (ignoring the sign for now). From the 45-45-90 triangle:

$$\theta_{\text{ref}} = \frac{\pi}{4}$$

Step 2: Determine which quadrants apply.

Tangent is negative in Quadrants II and IV.

Step 3: Find both angles in $[0, 2\pi]$.

Quadrant II:

$$x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Quadrant IV:

$$x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

Step 4: Write the general solution.

Since tangent repeats every π, these two angles are actually one family separated by exactly π:

$$\frac{7\pi}{4} - \frac{3\pi}{4} = \frac{4\pi}{4} = \pi \quad \checkmark$$

So the general solution can be written as a single family:

$$x = \frac{3\pi}{4} + \pi n, \qquad n \in \mathbb{Z}$$

Step 5: Verify with $n = 0$ and $n = 1$.

$$x = \frac{3\pi}{4}: \quad \tan\left(\frac{3\pi}{4}\right) = -1 \quad \checkmark$$

$$x = \frac{3\pi}{4} + \pi = \frac{7\pi}{4}: \quad \tan\left(\frac{7\pi}{4}\right) = -1 \quad \checkmark$$

$$\boxed{x = \frac{3\pi}{4} + \pi n, \qquad n \in \mathbb{Z}}$$

Common Mistakes

- **Forgetting the second solution.** Sine and cosine are each positive in two quadrants and negative in two quadrants. Most equations have two solutions per period. Always check both quadrants before moving on.

- **Not expanding the interval for $\sin(bx)$ problems.** If the equation contains $\sin(2x)$, you must find all values of the inner expression $2x$ in an interval twice as wide as the original. Skipping this step causes you to miss half the solutions.

- **Mixing up Quadrant II and III formulas.**

  - Quadrant II: $\pi - \theta_{\text{ref}}$
  - Quadrant III: $\pi + \theta_{\text{ref}}$

  These look similar but give very different angles.

- **Using $2\pi n$ for tangent general solutions.** Tangent repeats every π, not 2π. The general solution for tangent uses πn, not $2\pi n$.

- **Treating $\sin^2(x)$ problems as linear.** If the equation is quadratic in the trig function, you must factor or use the quadratic formula. You cannot simply take a square root of both sides unless the equation is in the form $\sin^2(x) = k$.

Worksheet

Solve each equation on the interval $[0, 2\pi]$ unless stated otherwise.

1. $\sin(x) = \dfrac{\sqrt{3}}{2}$

2. $\cos(x) = -\dfrac{\sqrt{2}}{2}$

3. $2\sin(x) + \sqrt{3} = 0$

4. $2\cos^2(x) + \cos(x) - 1 = 0$

5. $\sin(2x) = \dfrac{1}{2}$

6. Find the **general solution** for $\tan(x) = \sqrt{3}$.

Answer Key

1. $\sin(x) = \dfrac{\sqrt{3}}{2}$ on $[0, 2\pi]$

Step 1: Find the reference angle.

We need an angle whose sine equals $\dfrac{\sqrt{3}}{2}$. From the 30-60-90 triangle:

$$\theta_{\text{ref}} = \frac{\pi}{3}$$

Step 2: Sine is positive, so use Quadrants I and II.

Quadrant I:

$$x = \frac{\pi}{3}$$

Quadrant II:

$$x = \pi - \frac{\pi}{3} = \frac{3\pi}{3} - \frac{\pi}{3} = \frac{2\pi}{3}$$

Step 3: Verify.

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \quad \checkmark \qquad \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} \quad \checkmark$$

$$\boxed{x = \frac{\pi}{3} \quad \text{and} \quad x = \frac{2\pi}{3}}$$

2. $\cos(x) = -\dfrac{\sqrt{2}}{2}$ on $[0, 2\pi]$

Step 1: Find the reference angle.

We need an angle whose cosine equals $\dfrac{\sqrt{2}}{2}$ (ignoring the sign). From the 45-45-90 triangle:

$$\theta_{\text{ref}} = \frac{\pi}{4}$$

Step 2: Cosine is negative, so use Quadrants II and III.

Quadrant II:

$$x = \pi - \frac{\pi}{4} = \frac{4\pi}{4} - \frac{\pi}{4} = \frac{3\pi}{4}$$

Quadrant III:

$$x = \pi + \frac{\pi}{4} = \frac{4\pi}{4} + \frac{\pi}{4} = \frac{5\pi}{4}$$

Step 3: Verify.

$$\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} \quad \checkmark \qquad \cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} \quad \checkmark$$

$$\boxed{x = \frac{3\pi}{4} \quad \text{and} \quad x = \frac{5\pi}{4}}$$

3. $2\sin(x) + \sqrt{3} = 0$ on $[0, 2\pi]$

Step 1: Isolate the trig function.

$$2\sin(x) = -\sqrt{3}$$

$$\sin(x) = -\frac{\sqrt{3}}{2}$$

Step 2: Find the reference angle.

We need an angle whose sine equals $\frac{\sqrt{3}}{2}$ (ignoring the sign):

$$\theta_{\text{ref}} = \frac{\pi}{3}$$

Step 3: Sine is negative, so use Quadrants III and IV.

Quadrant III:

$$x = \pi + \frac{\pi}{3} = \frac{3\pi}{3} + \frac{\pi}{3} = \frac{4\pi}{3}$$

Quadrant IV:

$$x = 2\pi - \frac{\pi}{3} = \frac{6\pi}{3} - \frac{\pi}{3} = \frac{5\pi}{3}$$

Step 4: Verify.

$$\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2} \quad \checkmark \qquad \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2} \quad \checkmark$$

$$\boxed{x = \frac{4\pi}{3} \quad \text{and} \quad x = \frac{5\pi}{3}}$$

4. $2\cos^2(x) + \cos(x) - 1 = 0$ on $[0, 2\pi]$

Step 1: Let $u = \cos(x)$ to reveal the quadratic structure.

$$2u^2 + u - 1 = 0$$

Step 2: Factor.

$$(2u - 1)(u + 1) = 0$$

Step 3: Solve for u.

$$2u - 1 = 0 \quad \Rightarrow \quad u = \frac{1}{2} \qquad \text{or} \qquad u + 1 = 0 \quad \Rightarrow \quad u = -1$$

So $\cos(x) = \frac{1}{2}$ or $\cos(x) = -1$.

Step 4: Solve $\cos(x) = \frac{1}{2}$.

Reference angle: $\theta_{\text{ref}} = \frac{\pi}{3}$. Cosine is positive in Quadrants I and IV:

$$x = \frac{\pi}{3} \qquad \text{and} \qquad x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

Step 5: Solve $\cos(x) = -1$.

Cosine equals -1 only at the leftmost point of the unit circle:

$$x = \pi$$

Step 6: Verify all three solutions.

For $x = \frac{\pi}{3}$:

$$2\left(\frac{1}{2}\right)^2 + \frac{1}{2} - 1 = 2 \cdot \frac{1}{4} + \frac{1}{2} - 1 = \frac{1}{2} + \frac{1}{2} - 1 = 0 \quad \checkmark$$

For $x = \pi$:

$$2(-1)^2 + (-1) - 1 = 2 - 1 - 1 = 0 \quad \checkmark$$

For $x = \frac{5\pi}{3}$: same as $x = \frac{\pi}{3}$ since $\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$.

$$0 \quad \checkmark$$

$$\boxed{x = \frac{\pi}{3}, \quad x = \pi, \quad x = \frac{5\pi}{3}}$$

5. $\sin(2x) = \frac{1}{2}$ on $[0, 2\pi]$

Step 1: Let $u = 2x$ and determine the new interval.
If $x \in [0, 2\pi]$, then:

$$u = 2x \in [0, 4\pi]$$

We must find all solutions to $\sin(u) = \frac{1}{2}$ on $[0, 4\pi]$.

Step 2: Find the reference angle.

$$u_{\text{ref}} = \frac{\pi}{6}$$

Step 3: Sine is positive, so use Quadrants I and II. Find all values in $[0, 4\pi]$.
First period $[0, 2\pi]$:

$$u = \frac{\pi}{6} \qquad \text{and} \qquad u = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Second period $[2\pi, 4\pi]$ (add 2π to each):

$$u = \frac{\pi}{6} + 2\pi = \frac{\pi}{6} + \frac{12\pi}{6} = \frac{13\pi}{6} \qquad \text{and} \qquad u = \frac{5\pi}{6} + 2\pi = \frac{5\pi}{6} + \frac{12\pi}{6} = \frac{17\pi}{6}$$

Step 4: Divide all u values by 2 to get x.

$$x = \frac{u}{2}$$

$$x = \frac{\pi/6}{2} = \frac{\pi}{12} \qquad x = \frac{5\pi/6}{2} = \frac{5\pi}{12} \qquad x = \frac{13\pi/6}{2} = \frac{13\pi}{12} \qquad x = \frac{17\pi/6}{2} = \frac{17\pi}{12}$$

Step 5: Verify all four solutions.

$$\sin\left(2 \cdot \frac{\pi}{12}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \quad \checkmark$$

$$\sin\left(2 \cdot \frac{5\pi}{12}\right) = \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2} \quad \checkmark$$

$$\sin\left(2 \cdot \frac{13\pi}{12}\right) = \sin\left(\frac{13\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \quad \checkmark$$

$$\sin\left(2 \cdot \frac{17\pi}{12}\right) = \sin\left(\frac{17\pi}{6}\right) = \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2} \quad \checkmark$$

$$\boxed{x = \frac{\pi}{12}, \quad x = \frac{5\pi}{12}, \quad x = \frac{13\pi}{12}, \quad x = \frac{17\pi}{12}}$$

6. General solution for $\tan(x) = \sqrt{3}$

Step 1: Find the reference angle.
We need an angle whose tangent equals $\sqrt{3}$. From the 30-60-90 triangle:

$$\tan\left(\frac{\pi}{3}\right) = \frac{\sin(\pi/3)}{\cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

$$\theta_{\text{ref}} = \frac{\pi}{3}$$

Step 2: Tangent is positive, so it falls in Quadrants I and III.
In $[0, 2\pi]$, the two solutions are:

$$x = \frac{\pi}{3} \qquad \text{and} \qquad x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$$

Note that $\frac{4\pi}{3} - \frac{\pi}{3} = \pi$, confirming these are exactly one tangent period apart.

Step 3: Write the general solution.
Since tangent repeats every π, both solutions are captured by one family:

$$x = \frac{\pi}{3} + \pi n, \qquad n \in \mathbb{Z}$$

Step 4: Verify with $n = 0$ and $n = 1$.

$$n = 0: \quad x = \frac{\pi}{3}, \quad \tan\left(\frac{\pi}{3}\right) = \sqrt{3} \quad \checkmark$$

$$n = 1: \quad x = \frac{\pi}{3} + \pi = \frac{4\pi}{3}, \quad \tan\left(\frac{4\pi}{3}\right) = \sqrt{3} \quad \checkmark$$

$$\boxed{x = \frac{\pi}{3} + \pi n, \qquad n \in \mathbb{Z}}$$

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