Solving More Complex Trigonometric Equations Using Technology
ApexMath

Notes

In Lesson 9, every equation had a solution at a special angle — one you could read directly from the 30-60-90 or 45-45-90 triangle. Most real-world problems are not that clean. When the value on the right side of the equation is not a special ratio, you need a calculator to find the reference angle.

This lesson covers exactly that: how to combine algebraic steps with a calculator to solve equations that do not have exact answers at special angles. It also builds on the coefficient and quadratic techniques from Lesson 9, now applied to messier numbers.

The Core Strategy

The method is almost identical to Lesson 9. The only change is in Step 2: instead of reading the reference angle from a special triangle, you press the inverse trig button on your calculator.

Step 1. Isolate the trig function on one side of the equation.

Step 2. Use your calculator's inverse trig key to find the **reference angle** from the *absolute value* of the right-hand side:

- $\sin(x) = k$: $\quad \theta_{\text{ref}} = \arcsin(|k|)$
- $\cos(x) = k$: $\quad \theta_{\text{ref}} = \arccos(|k|)$
- $\tan(x) = k$: $\quad \theta_{\text{ref}} = \arctan(|k|)$

Step 3. Use CAST to identify which quadrants produce the correct sign.

Step 4. Build each solution using the quadrant formulas.

Step 5. Check that every solution falls inside the required interval.

Step 6. Verify by substituting back into the original equation.

Quadrant Formulas (Review)

Quadrant	**Formula**
I	$x = \theta_{\text{ref}}$
II	$x = \pi - \theta_{\text{ref}}$
III	$x = \pi + \theta_{\text{ref}}$
IV	$x = 2\pi - \theta_{\text{ref}}$

Which Quadrants to Use

- $\sin(x) > 0$: Quadrants **I and II**
- $\sin(x) < 0$: Quadrants **III and IV**
- $\cos(x) > 0$: Quadrants **I and IV**
- $\cos(x) < 0$: Quadrants **II and III**
- $\tan(x) > 0$: Quadrants **I and III**
- $\tan(x) < 0$: Quadrants **II and IV**

Calculator Settings

Always confirm your calculator is set to **radians** before computing any inverse trig value. An answer accidentally computed in degree mode will be completely wrong.

Round all decimal answers to **four decimal places** unless told otherwise.

Equations with a Coefficient on x

When the equation contains $\sin(bx)$ or $\cos(bx)$, use the same expanded-interval technique from Lesson 9:

Step 1. Let $u = bx$. The new interval for u is $[0,\ b \cdot 2\pi]$.

Step 2. Find all values of u in the expanded interval.

Step 3. Divide every u solution by b to recover x.

You should expect $2b$ solutions in total for a standard sine or cosine equation on $[0, 2\pi]$.

Quadratic Trig Equations with Mixed Solutions

Some quadratic equations factor into one root that is a special angle and one that requires a calculator. Solve each root separately and handle each one individually.

Using Graphing Technology

A graphing calculator or Desmos can help in two ways:

- **Before solving:** Graph both sides of the equation and count how many intersection points exist. This tells you how many solutions to expect.
- **After solving:** Plot your algebraic solutions on the same graph to confirm they match the intersection points.

Technology checks your work — it does not replace the algebraic steps.

Pro Tips

- **Take the inverse trig of the absolute value.** If $\cos(x) = -0.4$, compute $\arccos(0.4)$ to get the reference angle. Then use the quadrant formulas to build the actual solutions. Using the reference angle method gets both solutions cleanly every time.
- **The calculator gives one angle. You find the rest.** arcsin, arccos, and arctan each return only one value. That is your reference angle. You must still apply the quadrant formulas to find all solutions.
- **Expand the interval before solving $\sin(bx)$ problems.** If $x \in [0, 2\pi]$ and the equation contains $\sin(3x)$, the inner expression $3x$ lives in $[0, 6\pi]$. Expect 6 solutions. Dividing the interval is wrong — always *multiply* the upper bound by b.
- **Keep full precision until the final step.** Rounding the reference angle early and then using it in $\pi - \theta_{\text{ref}}$ or $\pi + \theta_{\text{ref}}$ introduces compounding error. Round only at the very end.
- **For quadratic equations, always factor first.** Most exam problems are designed to factor cleanly. Only reach for the quadratic formula if factoring fails.

Examples

Example 1. Solve $\sin(x) = 0.6$ on $[0, 2\pi]$.

Step 1: The trig function is already isolated.
Step 2: Find the reference angle.
The value 0.6 is positive, so:

$$\theta_{\text{ref}} = \arcsin(0.6) \approx 0.6435 \text{ rad}$$

Step 3: Sine is positive. Use Quadrants I and II.
Step 4: Build both solutions.
Quadrant I:

$$x_1 = \theta_{\text{ref}} \approx 0.6435$$

Quadrant II:

$$x_2 = \pi - \theta_{\text{ref}} \approx 3.1416 - 0.6435 \approx 2.4981$$

Step 5: Both values are in $[0, 2\pi]$**.** ✓
Step 6: Verify.

$$\sin(0.6435) \approx 0.6000 \quad \checkmark \qquad \sin(2.4981) \approx 0.6000 \quad \checkmark$$

$$\boxed{x \approx 0.6435 \quad \text{and} \quad x \approx 2.4981}$$

Example 2. Solve $\cos(x) = -0.4$ on $[0, 2\pi]$.

Step 1: The trig function is already isolated.
Step 2: Find the reference angle.
The value is negative, so take the absolute value first:

$$\theta_{\text{ref}} = \arccos(0.4) \approx 1.1593 \text{ rad}$$

Step 3: Cosine is negative. Use Quadrants II and III.
Step 4: Build both solutions.
Quadrant II:

$$x_1 = \pi - \theta_{\text{ref}} \approx 3.1416 - 1.1593 \approx 1.9823$$

Quadrant III:

$$x_2 = \pi + \theta_{\text{ref}} \approx 3.1416 + 1.1593 \approx 4.3009$$

Step 5: Both values are in $[0, 2\pi]$**.** ✓
Step 6: Verify.

$$\cos(1.9823) \approx -0.4000 \quad \checkmark \qquad \cos(4.3009) \approx -0.4000 \quad \checkmark$$

$$\boxed{x \approx 1.9823 \quad \text{and} \quad x \approx 4.3009}$$

Example 3. Solve $3\sin(2x) - 1 = 0$ on $[0, 2\pi]$.

Step 1: Isolate the trig function.

$$3\sin(2x) = 1 \qquad \Rightarrow \qquad \sin(2x) = \frac{1}{3}$$

Step 2: Let $u = 2x$ and expand the interval.
Since $x \in [0, 2\pi]$:

$$u = 2x \in [0, 4\pi]$$

Expect $2 \times 2 = 4$ solutions.

Step 3: Find the reference angle.

$$\theta_{\text{ref}} = \arcsin\left(\frac{1}{3}\right) \approx 0.3398 \text{ rad}$$

Step 4: Sine is positive. Use Quadrants I and II. Find all u values in $[0, 4\pi]$.
First period $[0,\ 2\pi]$:

$$u_1 = 0.3398 \qquad \text{and} \qquad u_2 = \pi - 0.3398 \approx 2.8018$$

Second period $[2\pi,\ 4\pi]$ — add $2\pi \approx 6.2832$ to each:

$$u_3 \approx 0.3398 + 6.2832 \approx 6.6230 \qquad \text{and} \qquad u_4 \approx 2.8018 + 6.2832 \approx 9.0849$$

Step 5: Divide every u value by 2 to get x.

$$x_1 = \frac{0.3398}{2} \approx 0.1699 \qquad x_2 = \frac{2.8018}{2} \approx 1.4009$$

$$x_3 = \frac{6.6230}{2} \approx 3.3115 \qquad x_4 = \frac{9.0849}{2} \approx 4.5425$$

Step 6: All four values are in $[0, 2\pi]$. ✓

Step 7: Spot-check two solutions.

$$3\sin(2 \times 0.1699) - 1 = 3\sin(0.3398) - 1 \approx 3(0.3333) - 1 = 0 \quad ✓$$

$$3\sin(2 \times 3.3115) - 1 = 3\sin(6.6230) - 1 \approx 3(0.3333) - 1 = 0 \quad ✓$$

$$\boxed{x \approx 0.1699, \quad 1.4009, \quad 3.3115, \quad 4.5425}$$

Example 4. Solve $2\cos^2(x) - 3\cos(x) + 1 = 0$ on $[0, 2\pi]$.

Step 1: Substitute $u = \cos(x)$ to reveal the quadratic structure.

$$2u^2 - 3u + 1 = 0$$

Step 2: Factor.

$$(2u - 1)(u - 1) = 0$$

Step 3: Solve for u.

$$u = \frac{1}{2} \qquad \text{or} \qquad u = 1$$

So $\cos(x) = \dfrac{1}{2}$ or $\cos(x) = 1$.

Step 4: Solve $\cos(x) = \dfrac{1}{2}$.

This is a special angle. Reference angle: $\theta_{\text{ref}} = \dfrac{\pi}{3}$. Cosine is positive in Quadrants I and IV:

$$x = \frac{\pi}{3} \approx 1.0472 \qquad \text{and} \qquad x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \approx 5.2360$$

Step 5: Solve $\cos(x) = 1$.

Cosine equals 1 at the rightmost point of the unit circle. On the closed interval $[0, 2\pi]$, this occurs at both endpoints:

$$x = 0 \qquad \text{and} \qquad x = 2\pi \approx 6.2832$$

Step 6: Verify all four solutions.

$$2(1)^2 - 3(1) + 1 = 0 \quad \checkmark \quad (x = 0 \text{ and } x = 2\pi)$$

$$2\left(\tfrac{1}{2}\right)^2 - 3\left(\tfrac{1}{2}\right) + 1 = \tfrac{1}{2} - \tfrac{3}{2} + 1 = 0 \quad \checkmark \quad \left(x = \tfrac{\pi}{3} \text{ and } x = \tfrac{5\pi}{3}\right)$$

$$\boxed{x = 0, \quad \frac{\pi}{3}, \quad \frac{5\pi}{3}, \quad 2\pi}$$

Common Mistakes

- **Using the calculator answer as the only solution.** Inverse trig returns exactly one value. That is your reference angle — you still need to apply the quadrant formulas to find every solution.

- **Wrong quadrant for negative cosine.** When $\cos(x)$ is negative, the solutions are in Quadrants II and III, using $\pi - \theta_{\text{ref}}$ and $\pi + \theta_{\text{ref}}$. Using $2\pi - \theta_{\text{ref}}$ lands in Quadrant IV where cosine is *positive* — the wrong sign entirely.

- **Not expanding the interval for bx problems.** If $x \in [0, 2\pi]$ and the equation has $\sin(3x)$, the inner expression lives in $[0, 6\pi]$. Stopping at $[0, 2\pi]$ for the inner expression causes you to miss most of the solutions.

- **Dividing the interval instead of expanding it.** For $\sin(3x)$, multiply the upper bound: $[0, 6\pi]$. Dividing to get $[0, \frac{2\pi}{3}]$ is the opposite of what you need and produces almost no solutions.
- **Calculator in degree mode.** Always verify the mode before you start.
- **Rounding too early.** Keep the full decimal value of θ_{ref} and only round the final answers.

Worksheet

Solve each equation on $[0, 2\pi]$. Round all decimal answers to four decimal places.

1. $5\sin(x) - 2 = 0$

2. $\cos(x) = -0.7$

3. $4\tan(x) + 5 = 0$

4. $2\sin(3x) + 1 = 0$

5. $6\cos^2(x) - \cos(x) - 1 = 0$

Answer Key

1. $5\sin(x) - 2 = 0$ on $[0, 2\pi]$

Step 1: Isolate the trig function.

$$5\sin(x) = 2 \qquad \Rightarrow \qquad \sin(x) = \frac{2}{5} = 0.4$$

Step 2: Find the reference angle.

$$\theta_{\text{ref}} = \arcsin(0.4) \approx 0.4115 \text{ rad}$$

Step 3: Sine is positive. Use Quadrants I and II.
Quadrant I:

$$x_1 = 0.4115$$

Quadrant II:

$$x_2 = \pi - 0.4115 \approx 3.1416 - 0.4115 \approx 2.7301$$

Step 4: Both values are in $[0, 2\pi]$. ✓
Step 5: Verify.

$$5\sin(0.4115) - 2 \approx 5(0.4000) - 2 = 0 \quad \checkmark$$
$$5\sin(2.7301) - 2 \approx 5(0.4000) - 2 = 0 \quad \checkmark$$

$$\boxed{x \approx 0.4115 \quad \text{and} \quad x \approx 2.7301}$$

2. $\cos(x) = -0.7$ on $[0, 2\pi]$

Step 1: The trig function is already isolated.
Step 2: Find the reference angle from the absolute value.

$$\theta_{\text{ref}} = \arccos(0.7) \approx 0.7954 \text{ rad}$$

Step 3: Cosine is negative. Use Quadrants II and III.
Quadrant II:

$$x_1 = \pi - \theta_{\text{ref}} \approx 3.1416 - 0.7954 \approx 2.3462$$

Quadrant III:

$$x_2 = \pi + \theta_{\text{ref}} \approx 3.1416 + 0.7954 \approx 3.9370$$

Step 4: Both values are in $[0, 2\pi]$. ✓
Step 5: Verify.

$$\cos(2.3462) \approx -0.7000 \quad \checkmark \qquad \cos(3.9370) \approx -0.7000 \quad \checkmark$$

$$\boxed{x \approx 2.3462 \quad \text{and} \quad x \approx 3.9370}$$

3. $4\tan(x) + 5 = 0$ on $[0, 2\pi]$

Step 1: Isolate the trig function.

$$4\tan(x) = -5 \qquad \Rightarrow \qquad \tan(x) = -\frac{5}{4} = -1.25$$

Step 2: Find the reference angle from the absolute value.

$$\theta_{\text{ref}} = \arctan(1.25) \approx 0.8961 \text{ rad}$$

Step 3: Tangent is negative. Use Quadrants II and IV.
Quadrant II:

$$x_1 = \pi - \theta_{\text{ref}} \approx 3.1416 - 0.8961 \approx 2.2455$$

Quadrant IV:

$$x_2 = 2\pi - \theta_{\text{ref}} \approx 6.2832 - 0.8961 \approx 5.3871$$

Step 4: Both values are in $[0, 2\pi]$. ✓
Step 5: Verify.

$$4\tan(2.2455) + 5 \approx 4(-1.25) + 5 = 0 \quad \checkmark$$

$$4\tan(5.3871) + 5 \approx 4(-1.25) + 5 = 0 \quad \checkmark$$

$$\boxed{x \approx 2.2455 \quad \text{and} \quad x \approx 5.3871}$$

4. $2\sin(3x) + 1 = 0$ on $[0, 2\pi]$

Step 1: Isolate the trig function.

$$2\sin(3x) = -1 \qquad \Rightarrow \qquad \sin(3x) = -\frac{1}{2}$$

Step 2: Let $u = 3x$ and expand the interval.

$$u = 3x \in [0,\ 6\pi]$$

Since $b = 3$, expect $2 \times 3 = 6$ solutions.

Step 3: Find the reference angle.

This is a special angle:

$$\theta_{\text{ref}} = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Step 4: Sine is negative. Use Quadrants III and IV. List all u values across three full periods in $[0, 6\pi]$.

First period $[0,\ 2\pi]$:

$$u_1 = \pi + \frac{\pi}{6} = \frac{7\pi}{6} \qquad \text{and} \qquad u_2 = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

Second period $[2\pi,\ 4\pi]$ — add 2π:

$$u_3 = \frac{7\pi}{6} + 2\pi = \frac{19\pi}{6} \qquad \text{and} \qquad u_4 = \frac{11\pi}{6} + 2\pi = \frac{23\pi}{6}$$

Third period $[4\pi,\ 6\pi]$ — add another 2π:

$$u_5 = \frac{7\pi}{6} + 4\pi = \frac{31\pi}{6} \qquad \text{and} \qquad u_6 = \frac{11\pi}{6} + 4\pi = \frac{35\pi}{6}$$

Step 5: Divide every u value by 3 to get x.

$$x_1 = \frac{7\pi}{18} \approx 1.2217 \qquad x_2 = \frac{11\pi}{18} \approx 1.9199$$

$$x_3 = \frac{19\pi}{18} \approx 3.3161 \qquad x_4 = \frac{23\pi}{18} \approx 4.0143$$

$$x_5 = \frac{31\pi}{18} \approx 5.4105 \qquad x_6 = \frac{35\pi}{18} \approx 6.1087$$

Step 6: All six values are in $[0, 2\pi] \approx [0,\ 6.2832]$. ✓

Step 7: Spot-check two solutions.

$$2\sin\left(3 \cdot \frac{7\pi}{18}\right) + 1 = 2\sin\left(\frac{7\pi}{6}\right) + 1 = 2\left(-\frac{1}{2}\right) + 1 = 0 \quad \checkmark$$

$$2\sin\left(3 \cdot \frac{35\pi}{18}\right) + 1 = 2\sin\left(\frac{35\pi}{6}\right) + 1 = 2\sin\left(\frac{11\pi}{6}\right) + 1 = 2\left(-\frac{1}{2}\right) + 1 = 0 \quad \checkmark$$

$$\boxed{x = \frac{7\pi}{18},\ \frac{11\pi}{18},\ \frac{19\pi}{18},\ \frac{23\pi}{18},\ \frac{31\pi}{18},\ \frac{35\pi}{18}}$$

5. $6\cos^2(x) - \cos(x) - 1 = 0$ on $[0, 2\pi]$

Step 1: Substitute $u = \cos(x)$.

$$6u^2 - u - 1 = 0$$

Step 2: Factor.

$$(2u - 1)(3u + 1) = 0$$

Step 3: Solve for u.

$$2u - 1 = 0 \;\Rightarrow\; u = \frac{1}{2} \qquad \text{or} \qquad 3u + 1 = 0 \;\Rightarrow\; u = -\frac{1}{3}$$

So $\cos(x) = \frac{1}{2}$ or $\cos(x) = -\frac{1}{3}$.

Step 4: Solve $\cos(x) = \frac{1}{2}$.

Special angle. Reference angle: $\frac{\pi}{3}$. Cosine is positive in Quadrants I and IV:

$$x = \frac{\pi}{3} \approx 1.0472 \qquad \text{and} \qquad x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \approx 5.2360$$

Step 5: Solve $\cos(x) = -\frac{1}{3}$.

Not a special angle — use a calculator.
Find the reference angle from the absolute value:

$$\theta_{\text{ref}} = \arccos\left(\frac{1}{3}\right) \approx 1.2310 \text{ rad}$$

Cosine is negative in Quadrants II and III:
Quadrant II:

$$x = \pi - 1.2310 \approx 3.1416 - 1.2310 \approx 1.9106$$

Quadrant III:

$$x = \pi + 1.2310 \approx 3.1416 + 1.2310 \approx 4.3726$$

Step 6: All four solutions are in $[0, 2\pi]$. ✓
Step 7: Verify the non-special solutions.

For $x \approx 1.9106$, where $\cos(x) = -\frac{1}{3}$:

$$6\left(-\frac{1}{3}\right)^2 - \left(-\frac{1}{3}\right) - 1 = 6 \cdot \frac{1}{9} + \frac{1}{3} - 1 = \frac{2}{3} + \frac{1}{3} - 1 = 0 \quad \checkmark$$

$$\boxed{x = \frac{\pi}{3} \approx 1.0472, \quad x \approx 1.9106, \quad x \approx 4.3726, \quad x = \frac{5\pi}{3} \approx 5.2360}$$

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