Special Triangles
ApexMath

Notes

Why Special Triangles?

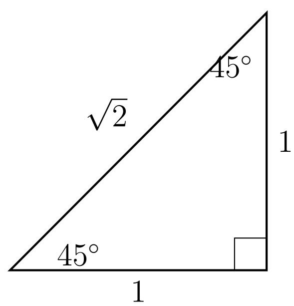
Most angles do not produce clean, exact trig values. However, two specific right triangles — the **45–45–90** triangle and the **30–60–90** triangle — produce exact values that appear constantly throughout trigonometry, calculus, and physics. You must be able to use both triangles fluently without a calculator.

The 45–45–90 Triangle

Start with a square of side length 1 and draw a diagonal. This creates a right triangle with two legs of length 1 and two base angles of 45°. Use the Pythagorean theorem to find the hypotenuse:

$$h = \sqrt{1^2 + 1^2} = \sqrt{2}$$

The three sides are in the ratio $1 : 1 : \sqrt{2}$.



The trig values for 45° are:

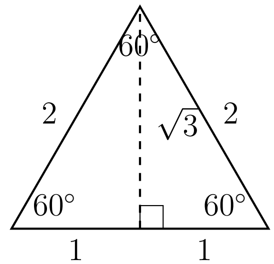
$$\sin 45^\circ = \frac{\sqrt{2}}{2} \qquad \cos 45^\circ = \frac{\sqrt{2}}{2} \qquad \tan 45^\circ = 1$$

The 30–60–90 Triangle

Start with an equilateral triangle of side length 2. Draw the height from one vertex to the opposite side. This splits it into two right triangles, each with a short leg of 1, a hypotenuse of 2, and a long leg:

$$\text{long leg} = \sqrt{2^2 - 1^2} = \sqrt{3}$$

The three sides are in the ratio $1 : \sqrt{3} : 2$.



The trig values for 30° and 60° are:

$$\sin 30° = \frac{1}{2} \qquad \sin 60° = \frac{\sqrt{3}}{2}$$

$$\cos 30° = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2}$$

$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \tan 60° = \sqrt{3}$$

Complete Reference Table

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	undefined

Finding All Six Ratios from One

If you are given one trig ratio, sketch a right triangle, label the two known sides, find the third using $a^2 + b^2 = c^2$, then read off all six ratios. The reciprocal functions are:

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Pro Tip

- **Sine increases from** 0° **to** 90°**:** 0, $\frac{1}{2}$, $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{3}}{2}$, 1. Cosine is the same list in reverse. Memorising this pattern lets you rebuild the entire table in seconds.

- **Always rationalise the denominator.** $\dfrac{1}{\sqrt{3}}$ is not fully simplified. Multiply top and bottom by $\sqrt{3}$ to get $\dfrac{\sqrt{3}}{3}$. An answer with a root in the denominator is incomplete.
- **SOH–CAH–TOA:** Sine = opposite/hypotenuse, Cosine = adjacent/hypotenuse, Tangent = opposite/adjacent. Always label your triangle before writing a ratio.
- **At $30°$ the angle is small, so the opposite side is short.** That is why $\sin 30° = \frac{1}{2}$ (small) and $\cos 30° = \frac{\sqrt{3}}{2}$ (large). At $60°$ they swap.

Examples

Example 1: Find the exact value of $\sin 30° + \cos 60°$.

Step 1: Look up both values from the reference table.

$$\sin 30° = \frac{1}{2} \qquad \cos 60° = \frac{1}{2}$$

Step 2: Add.

$$\sin 30° + \cos 60° = \frac{1}{2} + \frac{1}{2} = 1$$

Example 2: Find $\csc 45°$ and $\sec 45°$.

Step 1: Write the reciprocal definitions.

$$\csc 45° = \frac{1}{\sin 45°} \qquad \sec 45° = \frac{1}{\cos 45°}$$

Step 2: Substitute $\sin 45° = \cos 45° = \dfrac{\sqrt{2}}{2}$ and simplify.

$$\csc 45° = \frac{1}{\sqrt{2}/2} = \frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

By the same steps, $\sec 45° = \sqrt{2}$.

Example 3: Given $\sin\theta = \dfrac{3}{5}$ with θ in Quadrant I, find $\cos\theta$ and $\tan\theta$.

Step 1: Label the triangle. Opposite $= 3$, hypotenuse $= 5$.

Step 2: Find the adjacent side using the Pythagorean theorem.

$$\text{adj}^2 + 3^2 = 5^2 \quad \Rightarrow \quad \text{adj}^2 = 25 - 9 = 16 \quad \Rightarrow \quad \text{adj} = 4$$

Step 3: Read off the remaining ratios.

$$\cos\theta = \frac{4}{5} \qquad \tan\theta = \frac{3}{4}$$

Example 4: Find the exact value of $\tan^2 60° - \sin^2 30°$.
Step 1: Square each value first.

$$\tan 60° = \sqrt{3} \quad \Rightarrow \quad \tan^2 60° = 3$$

$$\sin 30° = \frac{1}{2} \quad \Rightarrow \quad \sin^2 30° = \frac{1}{4}$$

Step 2: Subtract.

$$3 - \frac{1}{4} = \frac{12}{4} - \frac{1}{4} = \frac{11}{4}$$

Common Mistakes

- **Confusing sine and cosine at $30°$ and $60°$.** At $30°$ the opposite side is short, so $\sin 30° = \frac{1}{2}$, not $\frac{\sqrt{3}}{2}$. Think about which side is opposite and which is adjacent before writing the ratio.
- **Not rationalising the denominator.** $\dfrac{1}{\sqrt{3}}$ is not a final answer. Multiply top and bottom by $\sqrt{3}$ to get $\dfrac{\sqrt{3}}{3}$.
- **Forgetting to square.** $\sin^2 30° = \left(\dfrac{1}{2}\right)^2 = \dfrac{1}{4}$, not $\dfrac{1}{2}$. Always apply the exponent to the entire fraction.
- **Using decimal approximations instead of exact values.** Unless the problem asks for a decimal, keep all answers in exact surd form.

Worksheet

1. Find the exact value of each expression without a calculator.

   (a) $\cos 30°$
   (b) $\tan 45°$
   (c) $\sin 60°$
   (d) $\cos 45° + \sin 45°$
   (e) $\tan 30° \cdot \tan 60°$

2. Find the exact value of each reciprocal trig function.

(a) $\csc 30^\circ$

(b) $\sec 60^\circ$

(c) $\cot 45^\circ$

(d) $\csc 60^\circ$

3. Given $\cos\theta = \dfrac{5}{13}$ with θ in Quadrant I, find all six trigonometric ratios.

4. Evaluate the following expressions exactly.

   (a) $\sin^2 45^\circ + \cos^2 45^\circ$

   (b) $2\sin 30^\circ \cdot \cos 30^\circ$

   (c) $\cos^2 60^\circ - \sin^2 60^\circ$

5. A right triangle has a 60° angle and a hypotenuse of length 10. Find the exact lengths of both legs.

Answer Key

1. Exact values.

(a) $\cos 30°$
From the 30–60–90 triangle, adjacent over hypotenuse at 30°:

$$\boxed{\cos 30° = \frac{\sqrt{3}}{2}}$$

(b) $\tan 45°$
From the 45–45–90 triangle, opposite over adjacent $= \dfrac{1}{1}$:

$$\boxed{\tan 45° = 1}$$

(c) $\sin 60°$
From the 30–60–90 triangle, opposite over hypotenuse at 60°:

$$\boxed{\sin 60° = \frac{\sqrt{3}}{2}}$$

(d) $\cos 45° + \sin 45°$
Step 1: Both values equal $\dfrac{\sqrt{2}}{2}$.

$$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

$$\boxed{\sqrt{2}}$$

(e) $\tan 30° \cdot \tan 60°$
Step 1: $\tan 30° = \dfrac{\sqrt{3}}{3}$ and $\tan 60° = \sqrt{3}$.

$$\frac{\sqrt{3}}{3} \cdot \sqrt{3} = \frac{3}{3} = 1$$

$$\boxed{1}$$

2. Reciprocal trig functions.

(a) $\csc 30°$
Step 1: Apply $\csc\theta = \dfrac{1}{\sin\theta}$.

$$\csc 30° = \frac{1}{1/2} = 2$$

$$\boxed{\csc 30° = 2}$$

(b) $\sec 60^\circ$

Step 1: Apply $\sec\theta = \dfrac{1}{\cos\theta}$.

$$\sec 60^\circ = \frac{1}{1/2} = 2$$

$$\boxed{\sec 60^\circ = 2}$$

(c) $\cot 45^\circ$

Step 1: Apply $\cot\theta = \dfrac{1}{\tan\theta}$.

$$\cot 45^\circ = \frac{1}{1} = 1$$

$$\boxed{\cot 45^\circ = 1}$$

(d) $\csc 60^\circ$

Step 1: Apply $\csc\theta = \dfrac{1}{\sin\theta}$.

$$\csc 60^\circ = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}$$

Step 2: Rationalise the denominator.

$$\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\boxed{\csc 60^\circ = \frac{2\sqrt{3}}{3}}$$

3. $\cos\theta = \dfrac{5}{13}$, Quadrant I.

Step 1: Adjacent $= 5$, hypotenuse $= 13$. Find the opposite side.

$$\text{opp}^2 = 13^2 - 5^2 = 169 - 25 = 144 \quad\Rightarrow\quad \text{opp} = 12$$

Step 2: Write all six ratios. All are positive in Quadrant I.

$$\sin\theta = \frac{12}{13} \qquad \cos\theta = \frac{5}{13} \qquad \tan\theta = \frac{12}{5}$$

$$\csc\theta = \frac{13}{12} \qquad \sec\theta = \frac{13}{5} \qquad \cot\theta = \frac{5}{12}$$

$$\boxed{\sin\theta = \frac{12}{13}, \quad \tan\theta = \frac{12}{5}, \quad \csc\theta = \frac{13}{12}, \quad \sec\theta = \frac{13}{5}, \quad \cot\theta = \frac{5}{12}}$$

4. Expressions.

(a) $\sin^2 45^\circ + \cos^2 45^\circ$

$$\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$$

$$\boxed{1}$$

This is the Pythagorean Identity $\sin^2\theta + \cos^2\theta = 1$, which holds for every angle.
(b) $2\sin 30° \cdot \cos 30°$

$$2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

$$\boxed{\frac{\sqrt{3}}{2}}$$

This equals $\sin 60°$, a preview of the double angle identity $\sin(2\theta) = 2\sin\theta\cos\theta$.
(c) $\cos^2 60° - \sin^2 60°$

$$\left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} - \frac{3}{4} = -\frac{2}{4} = -\frac{1}{2}$$

$$\boxed{-\frac{1}{2}}$$

This equals $\cos 120°$, a preview of the double angle identity $\cos(2\theta) = \cos^2\theta - \sin^2\theta$.

5. Right triangle with a $60°$ angle and hypotenuse $= 10$.
Step 1: The sides of a 30–60–90 triangle are in the ratio $1 : \sqrt{3} : 2$. The hypotenuse corresponds to 2 in the ratio.
Step 2: Find the scale factor.

$$\text{Scale factor} = \frac{10}{2} = 5$$

Step 3: Multiply each part of the ratio by 5.

$$\text{Short leg (opposite } 30°) = 1 \times 5 = 5$$

$$\text{Long leg (opposite } 60°) = \sqrt{3} \times 5 = 5\sqrt{3}$$

$$\boxed{\text{Short leg} = 5, \qquad \text{Long leg} = 5\sqrt{3}}$$

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