Trigonometric Identities — Sum, Difference, Double Angle, and the Unit Circle
ApexMath

Notes

A **trigonometric identity** is an equation that is true for every value of the variable where both sides are defined. Unlike a regular equation, you are not solving for x — you are proving or using a relationship that is always true.

Identities are used to simplify expressions, evaluate exact values at non-standard angles, and transform equations into a more useful form.

Part 1: The Pythagorean Identities

These three identities come directly from the unit circle and the Pythagorean theorem. They are the foundation everything else builds on.

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

The first one is the most important. The second and third are derived from it by dividing through by $\cos^2\theta$ or $\sin^2\theta$ respectively.

Useful rearrangements of the first identity:

$$\sin^2\theta = 1 - \cos^2\theta \qquad \text{and} \qquad \cos^2\theta = 1 - \sin^2\theta$$

Whenever you see $\sin^2\theta$ or $\cos^2\theta$ alone in an expression, these rearrangements let you swap one for the other.

Part 2: The Unit Circle Connection

The unit circle is a circle of radius 1 centred at the origin. For any angle θ, the point on the unit circle is:

$$(\cos\theta,\ \sin\theta)$$

This means the x-coordinate is the cosine and the y-coordinate is the sine.

Key unit circle values to know:

Angle	$\boldsymbol{\theta}$ (radians)	$\cos\theta$	$\sin\theta$
$0°$	0	1	0
$30°$	$\pi/6$	$\sqrt{3}/2$	$1/2$
$45°$	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$60°$	$\pi/3$	$1/2$	$\sqrt{3}/2$
$90°$	$\pi/2$	0	1
$180°$	π	-1	0
$270°$	$3\pi/2$	0	-1
$360°$	2π	1	0

For angles in Quadrants II, III, and IV, find the reference angle first and then apply the correct sign from CAST.

Part 3: Sum and Difference Identities

These identities let you find exact values for angles that are sums or differences of two familiar angles.

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

A memory trick for the signs:

- **Sine** always has the **same** sign as the original: $\sin(A \pm B)$ uses $\pm$ in the formula.
- **Cosine** always has the **opposite** sign: $\cos(A+B)$ uses subtraction, and $\cos(A-B)$ uses addition.

Part 4: Double Angle Identities

Double angle identities are a special case of the sum identities where $A = B$. They give exact formulas for $\sin(2\theta)$ and $\cos(2\theta)$.

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

For cosine, there are three equivalent forms — use whichever fits the problem best:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\cos(2\theta) = 2\cos^2\theta - 1$$
$$\cos(2\theta) = 1 - 2\sin^2\theta$$

All three are correct. The second form is useful when you want to eliminate $\sin^2\theta$. The third form is useful when you want to eliminate $\cos^2\theta$.

Part 5: Proving Identities

To prove that an identity is true, you work on **one side only** and transform it until it matches the other side. You do not move terms across the equals sign.

General strategy:

1. Start with the more complicated side.

2. Convert everything to sines and cosines if you are stuck.

3. Look for a Pythagorean identity substitution.

4. Factor or expand as needed.

5. Keep going until one side matches the other exactly.

Pro Tips

- **Memorize the four sum/difference formulas as a group.** Notice that sine uses mixed products ($\sin\cos$ and $\cos\sin$), while cosine uses matched products ($\cos\cos$ and $\sin\sin$). The only thing that changes is the sign in the middle.
- **For double angle cosine, choose the form that cancels the function you don't want.** If the problem only has sine, use $\cos(2\theta) = 1 - 2\sin^2\theta$ to write everything in terms of sine.
- **When proving identities, never cross the equals sign.** Work on the left side alone, or the right side alone. Changing both sides at once is not a valid proof.
- **Find cosine from sine (or vice versa) using Pythagoras.** If you are given $\sin\theta$ and need $\cos\theta$, use $\cos^2\theta = 1 - \sin^2\theta$ and choose the correct sign based on the quadrant.
- **For sum/difference problems, split the angle into two familiar pieces.** Angles like 75°, 105°, 15°, and $\frac{5\pi}{12}$ are all sums or differences of 30°, 45°, and 60°.

Examples

Example 1. Find the exact value of $\sin(75^\circ)$.

Step 1: Split 75° into two familiar angles.

$$75^\circ = 45^\circ + 30^\circ$$

Step 2: Apply the sine sum identity.

$$\sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

Step 3: Substitute the known values.

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

Step 4: Simplify.

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$\boxed{\sin(75^\circ) = \frac{\sqrt{6} + \sqrt{2}}{4}}$$

Example 2. Find the exact value of $\cos(15^\circ)$.

Step 1: Split 15° into two familiar angles.

$$15^\circ = 45^\circ - 30^\circ$$

Step 2: Apply the cosine difference identity.

$$\cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$

Step 3: Substitute the known values.

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

Step 4: Simplify.

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$\boxed{\cos(15^\circ) = \frac{\sqrt{6} + \sqrt{2}}{4}}$$

Example 3. Given $\sin A = \dfrac{3}{5}$ with A in Quadrant I, find $\sin(2A)$ and $\cos(2A)$.

Step 1: Find $\cos A$ using the Pythagorean identity.

$$\cos^2 A = 1 - \sin^2 A = 1 - \frac{9}{25} = \frac{16}{25}$$

$$\cos A = \frac{4}{5} \qquad \text{(positive because } A \text{ is in Quadrant I)}$$

Step 2: Apply the double angle identity for sine.

$$\sin(2A) = 2\sin A\cos A = 2 \cdot \frac{3}{5} \cdot \frac{4}{5} = \frac{24}{25}$$

Step 3: Apply the double angle identity for cosine.

$$\cos(2A) = \cos^2 A - \sin^2 A = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$$

Step 4: Verify using a second form of the cosine double angle identity.

$$\cos(2A) = 1 - 2\sin^2 A = 1 - 2 \cdot \frac{9}{25} = 1 - \frac{18}{25} = \frac{7}{25} \quad \checkmark$$

$$\boxed{\sin(2A) = \frac{24}{25} \qquad \cos(2A) = \frac{7}{25}}$$

Example 4. Simplify $\dfrac{\sin^2 x - 1}{\cos x}$.

Step 1: Recognise the numerator.
The Pythagorean identity tells us $\sin^2 x + \cos^2 x = 1$, so:

$$\sin^2 x - 1 = -(1 - \sin^2 x) = -\cos^2 x$$

Step 2: Substitute.

$$\frac{\sin^2 x - 1}{\cos x} = \frac{-\cos^2 x}{\cos x}$$

Step 3: Cancel one factor of $\cos x$.

$$= -\cos x$$

$$\boxed{\frac{\sin^2 x - 1}{\cos x} = -\cos x}$$

Example 5. Prove the identity: $\dfrac{1-\cos(2x)}{\sin(2x)} = \tan x$.

We will work on the **left side only** and transform it into $\tan x$.

Step 1: Replace $\cos(2x)$ using the double angle identity $\cos(2x) = 1 - 2\sin^2 x$.

$$\frac{1-\cos(2x)}{\sin(2x)} = \frac{1-(1-2\sin^2 x)}{\sin(2x)}$$

Step 2: Simplify the numerator.

$$= \frac{1-1+2\sin^2 x}{\sin(2x)} = \frac{2\sin^2 x}{\sin(2x)}$$

Step 3: Replace $\sin(2x)$ using the double angle identity $\sin(2x) = 2\sin x\cos x$.

$$= \frac{2\sin^2 x}{2\sin x\cos x}$$

Step 4: Cancel the common factor of $2\sin x$.

$$= \frac{\sin x}{\cos x}$$

Step 5: Recognise the result.

$$= \tan x \qquad \checkmark$$

The left side equals the right side, so the identity is proven. ■

Common Mistakes

- **Thinking $\sin(A+B) = \sin A + \sin B$.** This is false. Sine does not distribute over addition. You must use the full sum identity: $\sin(A+B) = \sin A\cos B + \cos A\sin B$.
- **Getting the cosine sum/difference signs backwards.** Remember: cosine has the *opposite* sign. $\cos(A+B)$ subtracts, and $\cos(A-B)$ adds. Many students flip these.
- **Forgetting the quadrant when finding cosine from sine.** If $\sin\theta = \frac{3}{5}$, then $\cos\theta$ is either $+\frac{4}{5}$ or $-\frac{4}{5}$ depending on the quadrant. Always use the given quadrant information before assigning a sign.
- **Using $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ when a simpler form would eliminate extra terms.** If the problem only has sines, use $\cos(2\theta) = 1 - 2\sin^2\theta$. Choosing the wrong form creates extra work.
- **Crossing the equals sign when proving identities.** The most common proof error is writing something like "LHS = RHS ⇒ (rearranged form)". You must transform one side into the other without touching the other side at all.
- **Cancelling terms instead of factors.** $\dfrac{\sin x + \cos x}{\sin x} \neq 1 + \cos x$. You can only cancel a factor that multiplies the entire numerator and denominator — not individual terms being added.

Worksheet

1. Find the exact value of $\sin(105°)$.

2. Find the exact value of $\cos(75°)$.

3. Given $\sin A = -\frac{5}{13}$ with A in Quadrant III, find $\sin(2A)$ and $\cos(2A)$.

4. Simplify $\cos^2 x - \sin^2 x$. Write your answer as a single trig function.

5. Prove the identity:

$$\frac{1 - \cos(2x)}{\sin(2x)} = \tan x$$

Hint: Start with the left side. Replace $\cos(2x)$ *and* $\sin(2x)$ *using the double angle identities.*

Answer Key

1. $\sin(105°)$

Step 1: Split $105°$ into two familiar angles.

$$105° = 60° + 45°$$

Step 2: Apply the sine sum identity.

$$\sin(60° + 45°) = \sin 60° \cos 45° + \cos 60° \sin 45°$$

Step 3: Substitute known values.

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$$

Step 4: Simplify.

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$\boxed{\sin(105°) = \frac{\sqrt{6} + \sqrt{2}}{4}}$$

2. $\cos(75°)$

Step 1: Split $75°$ into two familiar angles.

$$75° = 45° + 30°$$

Step 2: Apply the cosine sum identity.

$$\cos(45° + 30°) = \cos 45° \cos 30° - \sin 45° \sin 30°$$

Step 3: Substitute known values.

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

Step 4: Simplify.

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$\boxed{\cos(75°) = \frac{\sqrt{6} - \sqrt{2}}{4}}$$

3. $\sin A = -\dfrac{5}{13}$, A in Quadrant III.

Step 1: Find $\cos A$.

$$\cos^2 A = 1 - \sin^2 A = 1 - \frac{25}{169} = \frac{144}{169}$$

In Quadrant III, cosine is *negative*:

$$\cos A = -\frac{12}{13}$$

Step 2: Find $\sin(2A)$.

$$\sin(2A) = 2\sin A\cos A = 2 \cdot \left(-\frac{5}{13}\right) \cdot \left(-\frac{12}{13}\right) = 2 \cdot \frac{60}{169} = \frac{120}{169}$$

Step 3: Find $\cos(2A)$.

$$\cos(2A) = \cos^2 A - \sin^2 A = \frac{144}{169} - \frac{25}{169} = \frac{119}{169}$$

Step 4: Verify using a second form.

$$\cos(2A) = 1 - 2\sin^2 A = 1 - 2 \cdot \frac{25}{169} = 1 - \frac{50}{169} = \frac{119}{169} \quad \checkmark$$

$$\boxed{\sin(2A) = \frac{120}{169} \qquad \cos(2A) = \frac{119}{169}}$$

4. Simplify $\cos^2 x - \sin^2 x$.

Step 1: Recognise the double angle identity for cosine.
Looking at the three forms of $\cos(2x)$:

$$\cos(2x) = \cos^2 x - \sin^2 x$$

The expression $\cos^2 x - \sin^2 x$ is exactly the right side of this identity.
Step 2: Replace directly.

$$\cos^2 x - \sin^2 x = \cos(2x)$$

$$\boxed{\cos^2 x - \sin^2 x = \cos(2x)}$$

5. Prove: $\dfrac{1-\cos(2x)}{\sin(2x)} = \tan x$

We work on the **left side only**.

Step 1: Replace $\cos(2x)$ using the identity $\cos(2x) = 1 - 2\sin^2 x$.

$$\frac{1-\cos(2x)}{\sin(2x)} = \frac{1-(1-2\sin^2 x)}{\sin(2x)}$$

Step 2: Simplify the numerator by distributing the negative sign.

$$= \frac{1-1+2\sin^2 x}{\sin(2x)} = \frac{2\sin^2 x}{\sin(2x)}$$

Step 3: Replace $\sin(2x)$ using the identity $\sin(2x) = 2\sin x\cos x$.

$$= \frac{2\sin^2 x}{2\sin x\cos x}$$

Step 4: Cancel the common factor of $2\sin x$ from the numerator and denominator.

$$= \frac{\sin x}{\cos x}$$

Step 5: Rewrite using the definition of tangent.

$$= \tan x \qquad \checkmark$$

The left side equals $\tan x$, which is the right side. The identity is proven. ■

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