The Coordinate Plane and the Unit Circle
ApexMath

Notes

What Is the Unit Circle?

The **unit circle** is a circle with radius $r = 1$ centred at the origin. Its equation comes directly from the Pythagorean theorem:

$$x^2 + y^2 = 1$$

Every point $(x,\, y)$ on the unit circle corresponds to an angle θ in standard position. Those coordinates are not random numbers — they are the cosine and sine of the angle:

$$x = \cos\theta \qquad y = \sin\theta$$

This single idea is the foundation of all trigonometry. Once you know the angle, you can read its cosine and sine directly from the circle.

The Six Trigonometric Functions

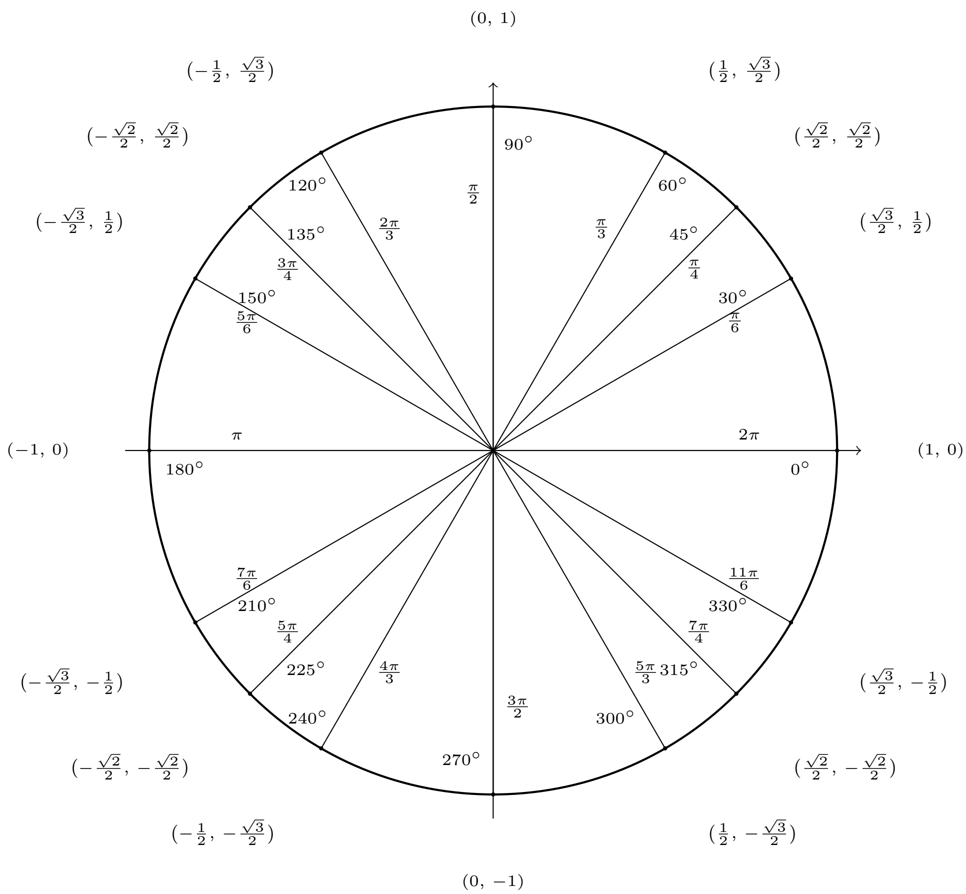
Using the point $(x,\, y)$ on the unit circle, all six trig functions are defined as:

$$\cos\theta = x \qquad \sin\theta = y \qquad \tan\theta = \frac{y}{x}, \quad x \neq 0$$

$$\sec\theta = \frac{1}{x}, \quad x \neq 0 \qquad \csc\theta = \frac{1}{y}, \quad y \neq 0 \qquad \cot\theta = \frac{x}{y}, \quad y \neq 0$$

The Unit Circle

The diagram below shows every key angle. Reading from outside in: the **coordinates** (x, y) sit outside the circle, the **degrees** sit just inside the rim, and the **radians** sit just inside the degrees.



Signs by Quadrant — ASTC

The sign of each trig function depends on which quadrant the terminal side falls in. A helpful memory tool is **"All Students Take Calculus"** (ASTC), read counterclockwise from Quadrant I. It tells you which functions are *positive*:

- **A**ll functions positive in Quadrant I
- **S**ine (and cosecant) positive in Quadrant II
- **T**angent (and cotangent) positive in Quadrant III
- **C**osine (and secant) positive in Quadrant IV

Finding Values Outside Quadrant I

Use the **reference angle** to find the magnitude of the trig value, then apply the ASTC sign rule to determine whether the result is positive or negative.

Pro Tip

- **Cosine is the x-coordinate; sine is the y-coordinate.** If you mix these up, every answer will be wrong. Repeat this until it is automatic.

- For the 30°–60° pair: at 30° the angle is small so sine is small $\left(\frac{1}{2}\right)$ and cosine is large $\left(\frac{\sqrt{3}}{2}\right)$. At 60° they swap. Use this reasoning to avoid confusing the two.
- $\cos\theta$ and $\sin\theta$ are **never** greater than 1 or less than -1, because they are coordinates on a circle of radius 1.
- $\tan\theta$ is undefined whenever $\cos\theta = 0$, i.e. at 90° and 270°. $\csc\theta$ is undefined whenever $\sin\theta = 0$, i.e. at 0° and 180°.

Examples

Example 1: Find $\cos\left(\dfrac{\pi}{3}\right)$ and $\sin\left(\dfrac{\pi}{3}\right)$.

Step 1: Recognise the angle. $\dfrac{\pi}{3} = 60°$.

Step 2: Read the unit circle. The point at $60°$ **is** $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$**.**

$$\cos\frac{\pi}{3} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Example 2: Find $\tan\left(\dfrac{\pi}{4}\right)$.

Step 1: Read the unit circle. $\dfrac{\pi}{4} = 45°$, point is $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$.

Step 2: Apply $\tan\theta = \dfrac{y}{x}$**.**

$$\tan\frac{\pi}{4} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$$

Example 3: Find the exact value of $\cos(150°)$.

Step 1: Find the quadrant and reference angle.

$$90° < 150° < 180° \quad \Rightarrow \quad \text{Quadrant II} \qquad \theta_{\text{ref}} = 180° - 150° = 30°$$

Step 2: Find the magnitude.

$$\cos(30°) = \frac{\sqrt{3}}{2}$$

Step 3: Apply ASTC. In Quadrant II, only sine is positive, so cosine is negative.

$$\cos(150°) = -\frac{\sqrt{3}}{2}$$

Example 4: Find $\sin(225°)$.

Step 1: Find the quadrant and reference angle.

$$180^\circ < 225^\circ < 270^\circ \quad \Rightarrow \quad \text{Quadrant III} \qquad \theta_{\text{ref}} = 225^\circ - 180^\circ = 45^\circ$$

Step 2: Find the magnitude.

$$\sin(45^\circ) = \frac{\sqrt{2}}{2}$$

Step 3: Apply ASTC. In Quadrant III, only tangent is positive, so sine is negative.

$$\sin(225^\circ) = -\frac{\sqrt{2}}{2}$$

Example 5: Find $\tan(300^\circ)$.
Step 1: Find the quadrant and reference angle.

$$270^\circ < 300^\circ < 360^\circ \quad \Rightarrow \quad \text{Quadrant IV} \qquad \theta_{\text{ref}} = 360^\circ - 300^\circ = 60^\circ$$

Step 2: Find the magnitudes at the reference angle.

$$\sin(60^\circ) = \frac{\sqrt{3}}{2} \qquad \cos(60^\circ) = \frac{1}{2}$$

Step 3: Apply ASTC. In Quadrant IV, only cosine is positive, so sine is negative.

$$\sin(300^\circ) = -\frac{\sqrt{3}}{2} \qquad \cos(300^\circ) = \frac{1}{2}$$

Step 4: Compute tangent.

$$\tan(300^\circ) = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$$

Common Mistakes

- **Swapping sine and cosine.** Cosine is the x-coordinate (horizontal) and sine is the y-coordinate (vertical). This is the single most common error in this entire lesson.
- **Forgetting the sign after finding the reference angle.** The reference angle gives you the magnitude only. You must still apply ASTC to decide the sign.
- **Mixing up $\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{3}}{2}$.** $\frac{\sqrt{2}}{2}$ always belongs to 45°. $\frac{\sqrt{3}}{2}$ always belongs to 30° or 60°.
- **Evaluating $\tan\theta$ at 90° or 270°.** At these points $\cos\theta = 0$, so $\tan\theta$ is undefined. There is no numerical answer.

Worksheet

1. Use the unit circle to find the exact value of each expression.

   (a) $\cos(0^\circ)$
   (b) $\sin(90^\circ)$
   (c) $\cos(180^\circ)$
   (d) $\sin(270^\circ)$
   (e) $\cos(360^\circ)$

2. Find the exact value of each expression.

   (a) $\sin\left(\frac{\pi}{6}\right)$
   (b) $\cos\left(\frac{\pi}{4}\right)$
   (c) $\tan\left(\frac{\pi}{3}\right)$
   (d) $\sin\left(\frac{3\pi}{2}\right)$

3. Use reference angles to find the exact value of each expression.

   (a) $\cos(120^\circ)$
   (b) $\sin(210^\circ)$
   (c) $\tan(315^\circ)$
   (d) $\cos\left(\frac{5\pi}{6}\right)$
   (e) $\sin\left(\frac{4\pi}{3}\right)$

4. A point on the unit circle has $\cos\theta = -\frac{1}{2}$ and the terminal side is in Quadrant III. Find $\sin\theta$ and $\tan\theta$.

5. For what values of θ in $[0°, 360°)$ is $\tan\theta$ undefined? Explain why.

Answer Key

1. Unit circle values at the axis angles.

(a) $\cos(0°)$
The unit circle point at $0°$ is $(1,\ 0)$. Cosine is the x-coordinate.

$$\boxed{\cos(0°) = 1}$$

(b) $\sin(90°)$
The unit circle point at $90°$ is $(0,\ 1)$. Sine is the y-coordinate.

$$\boxed{\sin(90°) = 1}$$

(c) $\cos(180°)$
The unit circle point at $180°$ is $(-1,\ 0)$.

$$\boxed{\cos(180°) = -1}$$

(d) $\sin(270°)$
The unit circle point at $270°$ is $(0,\ -1)$.

$$\boxed{\sin(270°) = -1}$$

(e) $\cos(360°)$
$360°$ completes one full rotation and returns to $(1,\ 0)$.

$$\boxed{\cos(360°) = 1}$$

2. Exact values from the unit circle.

(a) $\sin\left(\frac{\pi}{6}\right)$

Step 1: $\frac{\pi}{6} = 30°$. The unit circle point is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Step 2: Sine is the y-coordinate.

$$\sin\frac{\pi}{6} = \frac{1}{2}$$

$$\boxed{\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}}$$

(b) $\cos\left(\frac{\pi}{4}\right)$

Step 1: $\frac{\pi}{4} = 45°$. The unit circle point is $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Step 2: Cosine is the x-coordinate.

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\boxed{\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}}$$

(c) $\tan\left(\frac{\pi}{3}\right)$

Step 1: $\frac{\pi}{3} = 60°$. The unit circle point is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Step 2: Apply $\tan\theta = \frac{y}{x}$.

$$\tan\frac{\pi}{3} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

$$\boxed{\tan\left(\frac{\pi}{3}\right) = \sqrt{3}}$$

(d) $\sin\left(\frac{3\pi}{2}\right)$

Step 1: $\frac{3\pi}{2} = 270°$. The unit circle point is $(0, -1)$.

$$\sin\frac{3\pi}{2} = -1$$

$$\boxed{\sin\left(\frac{3\pi}{2}\right) = -1}$$

3. Values using reference angles.

(a) $\cos(120°)$

Step 1: $90° < 120° < 180°$, so this is in Quadrant II.

$$\theta_{\text{ref}} = 180° - 120° = 60°$$

Step 2: $\cos(60°) = \frac{1}{2}$.

Step 3: In Quadrant II, cosine is negative.

$$\cos(120°) = -\frac{1}{2}$$

$$\boxed{\cos(120°) = -\frac{1}{2}}$$

(b) $\sin(210°)$

Step 1: $180° < 210° < 270°$, so this is in Quadrant III.

$$\theta_{\text{ref}} = 210° - 180° = 30°$$

Step 2: $\sin(30°) = \frac{1}{2}$.

Step 3: In Quadrant III, sine is negative.

$$\sin(210^\circ) = -\frac{1}{2}$$

$$\boxed{\sin(210^\circ) = -\frac{1}{2}}$$

(c) $\tan(315^\circ)$
Step 1: $270^\circ < 315^\circ < 360^\circ$, so this is in Quadrant IV.

$$\theta_{\text{ref}} = 360^\circ - 315^\circ = 45^\circ$$

Step 2: $\tan(45^\circ) = 1$.
Step 3: In Quadrant IV, cosine is positive and sine is negative, so tangent is negative.

$$\tan(315^\circ) = -1$$

$$\boxed{\tan(315^\circ) = -1}$$

(d) $\cos\left(\frac{5\pi}{6}\right)$
Step 1: $\frac{5\pi}{6} = 150^\circ$, which is in Quadrant II.

$$\theta_{\text{ref}} = 180^\circ - 150^\circ = 30^\circ$$

Step 2: $\cos(30^\circ) = \frac{\sqrt{3}}{2}$.
Step 3: In Quadrant II, cosine is negative.

$$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$$

$$\boxed{\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}}$$

(e) $\sin\left(\frac{4\pi}{3}\right)$
Step 1: $\frac{4\pi}{3} = 240^\circ$, which is in Quadrant III.

$$\theta_{\text{ref}} = 240^\circ - 180^\circ = 60^\circ$$

Step 2: $\sin(60^\circ) = \frac{\sqrt{3}}{2}$.
Step 3: In Quadrant III, sine is negative.

$$\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$\boxed{\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}}$$

4. $\cos\theta = -\frac{1}{2}$, Quadrant III.

Step 1: Use $x^2 + y^2 = 1$ to find y. Substitute $x = -\frac{1}{2}$.

$$\left(-\frac{1}{2}\right)^2 + y^2 = 1 \qquad \Rightarrow \qquad \frac{1}{4} + y^2 = 1 \qquad \Rightarrow \qquad y^2 = \frac{3}{4} \qquad \Rightarrow \qquad y = \pm\frac{\sqrt{3}}{2}$$

Step 2: In Quadrant III, $y < 0$, so $\sin\theta = -\frac{\sqrt{3}}{2}$.

Step 3: Compute tangent.

$$\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}/2}{-1/2} = \sqrt{3}$$

$$\boxed{\sin\theta = -\frac{\sqrt{3}}{2}, \qquad \tan\theta = \sqrt{3}}$$

5. $\tan\theta = \frac{\sin\theta}{\cos\theta}$, so it is undefined wherever $\cos\theta = 0$.

Step 1: The x-coordinate on the unit circle equals zero at the top and bottom of the circle.

$$\cos\theta = 0 \quad \text{at} \quad \theta = 90^\circ \quad \text{and} \quad \theta = 270^\circ$$

$$\boxed{\theta = 90^\circ \text{ and } \theta = 270^\circ}$$

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